Lecture 1

Introduction. The goal of the course is to help you get better at writing and reading proofs. Since one learns by doing, we will do a lot of math, including construction of the rational numbers out of integers and real numbers out of rationals.

Warning: Sally sweeps some important details under the rug. I will try and point out when this happens.

An informal discussion of sets. We start by talking about sets. Informally a set is a collection of some (mathematical) objects, which are called elements of the set.

Notation 1.1. We write $x \in A$ to mean "x is an element of a set A."
We write $x \notin A$ to mean "x is not an element of a set A."

For example:

- $\mathbb{N}$ = the set of all natural numbers, $\mathbb{N} = \{1, 2, 3, 4, 5 \ldots\}$;
- $\mathbb{Z}$ = the set of all integers, $\mathbb{Z} = \{0, 1, -1, 2, -2, \ldots\}$;
- $\mathbb{Q}$ = the set of all rational numbers:
$$\mathbb{Q} = \left\{ \frac{p}{q} \mid p, q \in \mathbb{Z}, \, q \neq 0 \right\}.$$
- $\mathbb{R}$ is the set of all real numbers.

(The notation above is currently standard in North America and, possibly, Western Europe.) Elements of $\mathbb{N}$ are natural numbers. For example, $10^{23} \in \mathbb{N}$, that is, 10^{23} is an element of $\mathbb{N}$. $-1 \notin \mathbb{N}$, that is, -1 is not an element of $\mathbb{N}$. We will show that there is a real positive number x so that $x^2 = 2$. This number is usually denoted by $\sqrt{2}$. We will show that $\sqrt{2} \notin \mathbb{Q}$, that is, $\sqrt{2}$ is not an element of $\mathbb{Q}$, or, less formally, $\sqrt{2}$ is not rational.

We take the point of view that sets are determined by their elements. That is two sets A and B are equal (we write $A = B$) if and only if[1] they have the same elements:
$$A = B \Leftrightarrow (a \in A \Leftrightarrow a \in B).$$

Definition 1.2 (Empty set)**.** The empty set, denoted $\varnothing$, is the set with no elements.

Definition 1.3 (Subsets)**.** A set A is a subset of a set B (we write $A \subset B$ or $A \subseteq B$) if (and only if) any element in A is also an element of B:
$$a \in A \Rightarrow a \in B.$$
(Here and elsewhere $\Rightarrow$ is a shorthand for "implies."

Example 1.4. $\mathbb{N} \subseteq \mathbb{Z}$, $\mathbb{Z} \subseteq \mathbb{Q}$, $\mathbb{Q} \subseteq \mathbb{R}$.

Remark 1.5. For any set A the empty set $\varnothing$ is a subset of A. Why?

Remark 1.6. $A = B \iff (A \subseteq B$ and $B \subseteq A)$. (This is Exercise 1.2.2 in Sally's book.)

Exercise 1.7 (Exercise 1.2.1 in Sally's book.)**.** For a set A,
$$A \subseteq A$$

Proof. $A \subseteq A$ means: for all $a \in A$, $a \in A$, which is true. □

Warning: In math, "X or Y" means:
X, Y, or *both* X and Y.

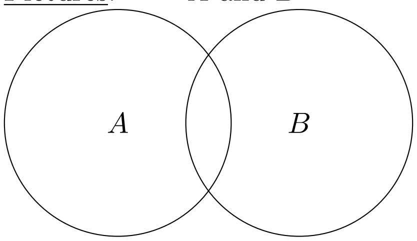
[1] We may abbreviate "if and only if" as $\Leftrightarrow$ or iff

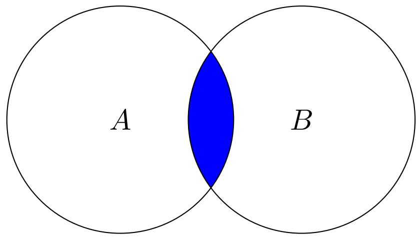
Definition 1.8 (union)**.** The union of two sets A and B is the set $A \cup B$ defined by
$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

Definition 1.9 (intersection)**.** The intersection of two sets A and B is the set $A \cap B$ defined by

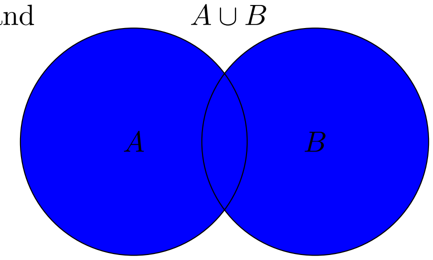
$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

<u>Pictures</u>: A and B $A \cap B$





and $A \cup B$



Example 1.10. $A = \{1, 2, 3\}, B = \{1, 2, 4\}$
$A \cup B = \{1, 2, 3, 4\}$
$A \cap B = \{1, 2\}$

Definition 1.11. Two sets A and B are disjoint if $A \cap B = \varnothing$.

Example 1.12. $A = \{1, 2, 3\}, B = \{4, 5, 6, 23\}$.
$A \cap B = \varnothing$
$\implies$ A and B are disjoint. $\{1, 2, 3\}$ and $\{2, 3, 6\}$ are *not* disjoint.

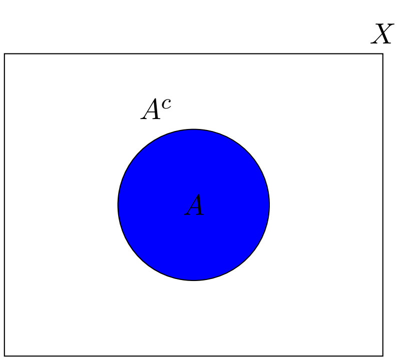
Example 1.13.

$$\{1, 2, 3\} \cap \{2, 4\} = \{2\}$$
$$\{1, 2, 3\} \cap \{4, 5\} = \varnothing$$

Definition 1.14. Two sets A and B are *disjoint* if (and only if) $A \cap B = \emptyset$. (That is, the sets A and B have no common elements.)

Definition 1.15 (Complements)**.** Let X be a set and A a subset of X. The *complement* of A in X is the set

$$A^c := \{x \in X \mid x \notin A\} :$$



Example 1.16. The complement of the set $\mathbb{N}$ of natural numbers inside the set $\mathbb{Z}$ of all integers is the set $\mathbb{N}^c = \{n \in \mathbb{Z} \mid n \notin \mathbb{N}\} = \{0, -1, -2, \ldots\}$.

Example 1.17. The complement of the set A of even integers inside the set X of a all integers is the set of all odd integers.

Example 1.18 (Example (1.3.7) in Sally)**.** For any three sets A, B and C

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

Proof. Suppose $x \in A \cap (B \cup C$. Then

$$x \in A \text{ and } x \in B \cup C.$$

Since $x \in B \cup C$ either $x \in B$ or $x \in C$ (or both). Now if $x \in B$ then $x \in A \cap B$ (since x is always in A). Similarly if $x \in C$ then $x \in A \cap C$. Hence $x \in (A \cap B) \cup (A \cap C)$. Since x is arbitrary we conclude that

$$A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C).$$

Conversely if $x \in (A \cap B) \cup (A \cap C)$ then either $x \in A \cap B$ (and then $x \in A$ and $x \in B$) or $x \in A \cap C$ (and then $x \in A$ and $x \in C$).

In either case $x \in A$ and ($x \in B$ or $x \in C$). Hence $x \in A$ and $x \in B \cup C$. Consequently $x \in A \cap (B \cup C$. It follows that

$$(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C).$$

By Remark **??** we conclude that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. □

Example 1.19 (1.3.7)**.** $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ for any sets A, B, C
Read the proof!

Example 1.20. Suppose A, B are two subsets of a set X. Then $(A \cup B)^c = A^c \cap B^c$.

Proof.

$$\begin{aligned}
x \in (A \cup B)^c &\iff x \in X \text{ and } x \notin (A \cup B) \\
&\iff x \in X \text{ and } (x \notin A \text{ and } x \notin B) \\
&\iff (x \in X \text{ and } x \notin A) \text{ and } (x \in X \text{ and } x \notin B) \\
&\iff x \in A^c \text{ and } x \in B^c \\
&\iff x \in A^c \cap B^c
\end{aligned}$$

□