Review problems for the first midterm exam, Math 347H **Prof. E. Lerman**
Monday, March 1 2021[1]

1 Give a brief answer (which is not necessarily a complete proof) to the following questions.

- Is it true that every finite set is countable?
- Is it true that a bijective map $f : A \to B$ is invertible?
- If $f : A \to B$, $g : B \to A$ are two functions such that $g(f(a)) = a$ for all $a \in A$ is f necessarily invertible?
- If $f : A \to B$, $g : B \to A$ are two injective functions is there a bijection $h : A \to B$?
- Is there a set A of the same cardinality as the power set $P(A)$ of A?
- Is it true that the set of natural numbers with the usual $+$ and $\cdot$ forms a ring?
- Is the set of rational numbers is countable?
- Can one deduce the principle of mathematical induction from the well-ordering principle? What do these two principles say?
- Is it true that the composition of functions is associative?
- Is it true that the composition of functions is commutative?
- Is $\mathbb{Z}_{10}$ an integral domain? Explain. What about $\mathbb{Z}_3$? Is $\mathbb{Z}_3$ an integral domain?

2 Prove that $2^n < n!$ for all natural numbers n greater than 3.

3 Prove that $n < 2^n$ for all $n \in \mathbb{N}$.

4

(1) Define what it means for A to be a set with $n > 0$ elements.
(2) Prove that if A has $n > 1$ elements and $a \in A$ then $B = A \smallsetminus \{a\}$ has $n - 1$ elements.
(3) Prove that if A has n elements and $f : A \to B$ is a bijection then B also has n elements.
(4) Suppose A and B are both countably infinite and $f : A \to B$ is onto. Does this imply that f is injective? Explain.
(5) Prove that if A and B are both sets with n elements ($n \in \mathbb{N}$) and $f : A \to B$ is onto then f is injective.

5 Let $f : A \to B$ be a function $\{U_i\}_{i \in I}$ be a family of subsets of B. Prove that

- $f^{-1}(\bigcup_{i \in I} U_i) = \bigcup_{i \in I} f^{-1}(U_i)$;
- $f^{-1}(\bigcap_{i \in I} U_i) = \bigcap_{i \in I} f^{-1}(U_i)$.

Suppose $X, Y \subset A$. Is it true that $f(X \cap Y) = f(X) \cap f(Y)$? Prove your answer.

6 Let $f : X \to Y$ be a function. Define a relation $\sim$ on X by

$$x \sim x' \Leftrightarrow f(x) = f(x').$$

- Prove that $\sim$ is an equivalence relation.
- What are the equivalence classes of this relation?
- Denote the set of equivalence classes of $\sim$ by $X/_\sim$. Prove that the map $\bar{f} : X/_\sim \to Y$ given by
$$\bar{f}([x]) := f(x)$$
is well-defined. Hint: what do you need to check?
- Prove that $\bar{f} : X/_\sim \to Y$ is injective.

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7 Let $\sim$ be an equivalence relation on a set X, $C, D \subset X$ two equivalence classes of $\sim$. Prove directly without quoting results in the lecture notes that if $D \cap C \neq \emptyset$ then $C = D$.

8 Recall that $\mathrm{Map}\,(X, Y) \equiv Y^X$ denotes the set of all functions from a set X to a set Y. Prove that there does not exist a surjective map from X to $\mathrm{Map}\,(X, \{0, 1\})$.
Hint: Do you remember the relation between $\mathrm{Map}\,(X, \{0, 1\})$ and the power set of X?

9 Prove that there is no bijection from $\mathbb{Q}$ to $\mathbb{R} \setminus \mathbb{Q}$.
Hint: the union of two countable sets is ...

10 Suppose X is a set, $\sim$ an equivalence relation and $X/\sim$ is the set of the equivalence classes.
(a) Prove that the function $f : X \to X/_\sim$, which takes $x \in X$ to its equivalence class, is onto. Hint: it's very short.
(b) Prove that for any $x, y \in X$

$$x \sim y \quad \Leftrightarrow \quad [x] = [y].$$

11 Prove that $f(x) = 3x$ defines a bijection from $\mathbb{R}$ to $\mathbb{R}$. Does it define a bijection from $\mathbb{Z}$ to $\mathbb{Z}$? If not, what goes wrong? If it does, what's the inverse?

12 Let A and B be two sets. Show that if A and B are countable then so are $A \cup B$ and $A \cap B$.

13 Is the set

$$C = \{(x, y) \in \mathbb{R}^2 \mid x^6 + y^4 = 1\}$$

the graph of a function from $\mathbb{R}$ to $\mathbb{R}$? Explain.

14 Prove that composition of functions is associative.

15 Prove that if $f : A \to B$ and $g : B \to C$ are both onto then so is their composite $g \circ f : A \to C$.

16 Suppose X is a set with three distinct elements. How many (distinct) equivalence relations are there on X?
Hint: count the number of distinct partitions of X.

17 There are 20 points within a $3m \times 3m$ square S. Show that some set of three of these points can be covered by a $1m \times 1m$ square. Hint: divide S into 9 $1m \times 1m$ squares.

18 Prove there exist two powers of 2 that differ by a multiple of 7777. Hint: Consider these powers of 2: 2^0, 2^1, 2^2, ..., 2^{7777} and consider their remainders after division by 7777.