Review for the final exam, Math 347H **Prof. E. Lerman**

Monday, May 3, 2021

Here is a brief (and probably incomplete) summary of the mathematics covered in the course.

- Equivalence relations and equivalence classes. Equivalence classes give rise to partitions.
- Well-ordering principle and mathematical induction.
- Division algorithm for integers and for polynomials with coefficients in a field.
- Construction of the field of rationals $\mathbb{Q}$ from the ring of integers $\mathbb{Z}$.
- Modular arithmetic, $\mathbb{Z}_n$.
- Injective, surjective and bijective maps.
- Pigeonhole principle. Finite and infinite sets.
- Schröder – Bernstein theorem
- Power set $\mathcal{P}(A)$ of a set A. There is no surjective map from A to $\mathcal{P}(A)$.
- Axiom of choice, countable and uncountable sets. Properties of countable sets.
- Ordered rings and fields
- Archimedean property of the reals.
- Cauchy sequences in rationals, reals and complex numbers, and in a metric space.
- Convergent sequences in rationals, reals, complex numbers and metric spaces.
- least upper bound property of the reals.
- Convergent sequences are Cauchy. Cauchy sequences are bounded. Cauchy sequences in $\mathbb{Q}$ form a ring.
- Series in $\mathbb{R}$ and in $\mathbb{C}$. Ratio test.
- Rings, fields, ring homomorphisms, ideals, quotient rings, first isomorphism theorem.
- Characteristic of a ring, rings of characteristic 0 contain a copy of $\mathbb{Z}$, fields of characteristic 0 contain a copy of $\mathbb{Q}$.
- Ordered rings are integral domains, have characteristic 0.
- Division algorithm for polynomials with coefficients in a field, complex numbers as a quotient ring.
- Vector spaces and linear maps.
- Norms on vector spaces. Cauchy-Schwarz inequality. Metrics from norms.
- Three equivalent norms on $\mathbb{R}^k$.
- The notion of a topology. Equivalent metrics define the same topology.
- Continuity — three equivalent definitions for maps between metric spaces. Continuity for maps between topological spaces.
- Closed and bounded sets in a metric space.
- Compact sets, Heine-Borel.
- Continuous maps take compact sets to compact sets.

(1) Suppose $\sim$ is an equivalence relation on a set X. Let $[x]$ denotes the equivalence class of $x \in X$ with respect to $\sim$. Prove that

$$x \sim y \qquad \Leftrightarrow \qquad [x] \cap [y] \neq \varnothing.$$

(2) Prove that a map $f : X \to Y$ is a bijection if and only if there is a map $g : Y \to X$ such that $f \circ g = id_Y$ and $g \circ f = id_X$.

(3) Let A be a nonempty set and $B \subset A$ a nonempty subset. Show that there is a surjective function $A \to B$.

(4) Prove that the collection of all sets cannot be a set.

(5) Suppose A is an infinite set and $f : A \to A$ is injective. Does f has to be onto? Prove your answer.

(6) Prove that a convergent sequence $\{x_n\}$ in a metric space is Cauchy.

(7) Let A and B be two sets. Suppose there are injections $f : A \to B$ and $g : B \to A$. Does there exist a bijection from A to B? Prove your answer.

(8) Suppose A and B are two countable sets. Prove that the union $A \cup B$ and product $A \times B$ are countable. State relevant theorems.

(9) Prove that the collection $\mathbb{Q}[x]$ of polynomials with rational coefficients is countable. Hint: what do you know about the size of a countable union of countable sets?

(10) Prove that the set $\mathbb{N}$ of natural numbers is the union of three disjoint countable infinite sets.

(11) Joe is proving by induction that $\sum_{i=0}^{n} q^i = \frac{q^{n+1}-1}{q-1}$. He assumes that the formula holds for $n = k$ and shows that it implies that the formula hold for $n = k + 1$. Is the proof complete or is there more to do?

(12) Suppose $\{a_n\}$ is a sequence of real numbers that converges to 10. Prove that there is $N \in \mathbb{N}$ so that for $n > N$ we have $a_n > 5$.

(13) Prove that that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

(14) Let F be a field. Then it is also a vector space over F. Prove directly from the definition of "linear" that if $T : F \to F$ is a linear map then there is $a \in F$ so that $T(x) = ax$ for all $x \in F$. Hint: for any $\lambda \in F$, $T(\lambda 1) = \lambda T(1)$.

(15) Prove that the field $\mathbb{C}$ of complex numbers could not be made into an ordered field.

(16) Recall that an integral domain is a commutative ring with unity (i.e., 1) and no zero divisors. Prove that if R is an integral domain then the ring of polynomials $R[x]$ has no zero divisors. Does $R[x]$ have a unity? Explain.

(17) Suppose F is a field, $p(x)$ is a polynomial with coefficients in F and $a \in F$ is a root of p. Prove that there is a polynomial $q(x)$ so that $p(x) = (x - a)q(x)$, that is, prove that $(x - a)$ divides p.

(18) Prove that if $\mathbb{Z}_n$ is a subring of a *field* then n is prime.

(19) Prove that $\mathbb{Z}_n$ cannot be a subring of an *ordered* field even if n is prime.

(20) Let $\{a_n\}$, $\{b_n\}$ be two Cauchy sequences in $\mathbb{R}$. Prove directly form the definition of a Cauchy sequence that $\{a_n + b_n\}$ is Cauchy.

(21) Prove that any convergent sequence in $\mathbb{R}$ is bounded. Use it to prove that if a sequence $\{a_n\}$ converges to L and $\{b_n\}$ converges to M then the product $\{a_n b_n\}$ converges to LM.

(22) Let $\{a_n\}$, $\{b_n\}$ be two sequences in $\mathbb{R}$. Suppose $\lim a_n = L$ and $\lim a_n b_n = LM$. Does it follow that $\{b_n\}$ converges to M? Explain. Hint: be careful!

(23) Let V be a vector space over the reals, $||\cdot||$ a norm on V. Suppose a sequence $x_n \to L$ and a sequence $y_n \to M$ in V. Prove that
  (a) $x_n + y_n \to L + M$.
  (b) For any $\lambda \in \mathbb{R}$, $\lambda x_n \to \lambda L$. Hint: be careful in the case $\lambda = 0$.

(24) Let R be a commutative ring, $\varphi : R \to S$ a ring homomorphism and I an ideal in R with $I \subset \ker \varphi$. Prove that

$$\tilde{\varphi} : R/I \to S \qquad \tilde{\varphi}(r + I) := \varphi(r)$$

is well-defined.

(25) Suppose a sequence of complex numbers $\{a_n\}$ converges to $L \neq 0$. Prove that the series $\sum a_n$ does not converge.

(26) Prove that an open ball $B_R(x)$ in a metric space (S, d) is open.

(27) Prove that an open set U in a metric space (S, d) is a union of open balls. Hint: for any $x \in U$ there is $R > 0$ such that ...

(28) Let $U \subset \mathbb{R}^k$ be an open set (with respect to the Euclidean metric). Prove that U is a union of countably many open balls. Hint: consider the collection $\mathcal{B}$ of all open balls in U that have the following two properties:
 (a) their radii are rational
 (b) the coordinates of their centers are rational.