Dec 2, 2005 Homework solutions.

Note Title 12/1/2005

§1 $\boxed{\#6}$ $\omega_{(3,-1,4)}(a_1,a_2,a_3) =$

$= \left(3^2(-1)\,dx + (-1)^2 4\,dy + 4^3 3\,dz\right)(a_1,a_2,a_3)$

$= -9\,a_1 + 4\,a_2 + 3\cdot 4^3\,a_3.$

$\boxed{\#7}$ $\omega_{(2,-1,-3,1)}(\vec{a},\vec{b}) =$

$\left(2\cdot(-3)\,dx_1\wedge dx_3 - (-1)\cdot 1\,dx_2\wedge dx_4\right)(\vec{a},\vec{b}) =$

$= -6\,(a_1 b_3 - b_1 a_3) + (a_2 b_4 - b_2 a_4)$

$\boxed{\#10}$ $(3\,dx + 2\,dy - x\,dz)\wedge(x^2 dx - \cos y\,dy + 7\,dz) =$

$= 3(-\cos y)\,dx\wedge dy + 3\cdot 7\,dx\wedge dz +$

$+ 2x^2\,dy\wedge dx + 2\cdot 7\,dy\wedge dz$

$+ (-x)\,dz\wedge(-\cos y\,dy) + (-x\,dz)\wedge(x^2 dx) =$

$= (-3\cos y - 2x^2)\,dx\wedge dy + (21 + x^3)\,dx\wedge dz +$

$+ (14 - x\cos y)\,dy\wedge dz$

$\boxed{\#11}$ $(y\,dx - x\,dy)\wedge(z\,dx\wedge dy + y\,dx\wedge dz + x\,dy\wedge dz) =$

$= xy\,dx\wedge dy\wedge dz - xy\,dy\wedge dx\wedge dz = 2xy\,dx\wedge dy\wedge dz.$

#16 A k-form in $\mathbb{R}^n$ looks like

$$\sum_{i_1,\dots,i_k=1}^{n} F_{i_1\dots i_k}\,dx_{i_1}\wedge\dots\wedge dx_{i_k}.$$

If $k>n$, then some of the indices $i_1, i_2, \dots, i_k$ have to repeat. But then

$$dx_{i_1}\wedge\dots\wedge dx_{i_k} = 0.$$

8.2 #6 $\int_{\vec{x}} \omega = \int_{[0,\pi]} \Big(b\, d(a\cos t) - a\, d(b\sin t) + (a\cos t)(b\sin t)\, d(ct) \Big) = \int_{[0,\pi]} \Big(ba(-\sin t)\, dt - ab\cos t\, dt +$

$+ abc\cos t\sin t\, dt \Big) = ba\cos t\Big|_0^{\pi} - ab\sin t\Big|_0^{\pi} + abc\int_0^{\pi} \cos t\sin t\, dt =$

$= ba(-1-1) - ab(0-0) + \frac{1}{2}abc\int_0^{\pi} \sin 2t\, dt = -2ab - \frac{1}{2}abc\frac{1}{2}\cos 2t\Big|_0^{\pi}$

$= -2ab$

#7 We parameterize the circle $\vec{x}(t) = (\cos t, \sin t)$, $0 \le t \le 2\pi$.

Then $\int_C \omega = \int_{[0,2\pi]} x^*\omega = \int_{[0,2\pi]} (\sin t\, d(\cos t) - \cos t\, d(\sin t)) =$

$= \int_{[0,2\pi]} (-\sin^2 t\, dt - \cos^2 t\, dt) = \int_{[0,2\pi]} (-dt) = -2\pi.$

#9 $\int_X \omega =$

$= \int_{[0,1]\times[0,4\pi]} t\, d(s\cos t)\wedge d(s\sin t) + 3d(t)\wedge d(s\cos t) - (s\cos t)\, d(s\sin t)\wedge dt$

$= \int_{[0,1]\times[0,4\pi]} t(\cos t\, ds - s\sin t\, dt)\wedge(\sin t\, ds + s\cos t\, dt) + 3\, dt\wedge(\cos t\, ds) -$

$- (s\cos t)\sin t\, ds\wedge dt) =$

$= \int_{[0,1]\times[0,4\pi]} st\cos^2 t\, ds\wedge dt + ts\sin^2 t\, ds\wedge dt + (-3\cos t)\, ds\wedge dt$

$- s\cos t\sin t\, ds\wedge dt$

$= \int_0^1\left(\int_0^{4\pi} \left(st - 3\cos t - \frac{s}{2}\sin 2t\right) dt\right) ds = \int_0^1 \left(s\frac{t^2}{2} - 3\sin t + \frac{s}{4}\cos 2t\right)\Big|_0^{4\pi} ds$

$= \int_0^1 s\frac{(4\pi)^2}{2}\, ds = \frac{16\pi^2}{2}\frac{s^2}{2}\Big|_0^1 = 4\pi^2$

#12

$$\int_S \omega = \iint_{x^2+y^2\le 4} e^{(x^2+y^2)}\, dx\wedge dy + y\, d(x^2+y^2)\wedge dx + x\, dy\wedge d(x^2+y^2) =$$

$$= \iint_{x^2+y^2\le 4} e^{x^2+y^2}\, dx\wedge dy + y\cdot 2y\, dy\wedge dx + x\, dy\wedge 2x\, dx$$

$$= \iint_{x^2+y^2\le 4} \left(e^{x^2+y^2} - 2y^2 - 2x^2\right) dx\, dy = \int_0^{2\pi}\int_0^2 \left(e^{r^2} - 2r^2\right) r\, dr\, d\theta =$$

$$= 2\pi \int_0^2 \left(e^{r^2} r - 2r^3\right) dr = 2\pi \left(e^{r^2} - \frac{2}{4} r^4\right)\Big|_0^1 = 2\pi\left(e - \frac{1}{2}\right).$$

#13

$$\int_X x_2\, dx_2\wedge dx_3\wedge dx_4 + 2x_1 x_3\, dx_1\wedge dx_2\wedge dx_3$$

$$= \int_{[0,1]\times[0,1]\times[0,1]} \left(u_2\, du_2\wedge du_3\wedge d(2u_1 - u_3)^2 + 2u_1u_3\, du_1\wedge du_2\wedge du_3\right) =$$

$$= \int_{[0,1]\times[0,1]\times[0,1]} u_2\, du_2\wedge du_3\wedge 2du_1 + 2u_1u_2\, du_1\wedge du_2\wedge du_3 =$$

$$= \int_0^1\int_0^1\int_0^1 (2u_2 + 2u_1u_2)\, du_3\, du_2\, du_1 = \int_0^1\int_0^1 2u_2(1+u_1)\, du_2\, du_1$$

$$= \int_0^1 \left(u^2\Big|_0^1\right)(1+u_1)\, du_1 = \left(u_1 + \frac{u_1^2}{2}\right)\Big|_0^1 = 3/2$$