Dec 9 HW SOLUTONS

Note Title 12/6/2005

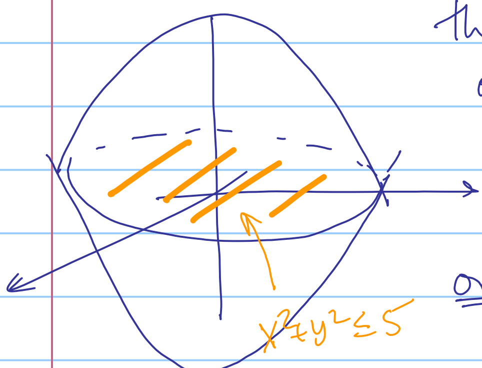
8.3 #4

$d(x_1 dx_2 - x_2 dx_1 + x_3 x_4 dx_4 - x_4 x_5 dx_5) = dx_1 \wedge dx_2 - dx_2 \wedge dx_1 + (x_4 dx_3 + x_3 dx_4) \wedge dx_4 - x_5 dx_4 \wedge dx_5 - x_4 dx_5 \wedge dx_5 =$
$= 2\, dx_1 \wedge dx_2 + x_4 dx_3 \wedge dx_4 - x_5 dx_4 \wedge dx_5.$

#5
$d(xz\, dx \wedge dy - y^2 z\, dx \wedge dz) = x\, dz \wedge dx \wedge dy - 2yz\, dy \wedge dx \wedge dz$
$= (x + 2yz)\, dx \wedge dy \wedge dz$

#6 $d(x_1 x_2 x_3\, dx_2 \wedge dx_3 \wedge dx_4 + x_2 x_3 x_4\, dx_1 \wedge dx_2 \wedge dx_3) =$
$= x_2 x_3\, dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 + x_2 x_3\, dx_4 \wedge dx_1 \wedge dx_2 \wedge dx_3) =$
$= (x_2 x_3 + (-1)^3 x_2 x_3)\, dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 = 0$

#10 $M = \{(x, y, z) \mid x^2 + y^2 - 6 \le z \le 4 - x^2 - y^2\}$. Note that it is the region in $\mathbb{R}^3$ trapped between the graphs of $z = x^2 + y^2 - 6$ and $z = 4 - x^2 - y^2$ over the region where $x^2 + y^2 - 6 \le 4 - x^2 - y^2$, i.e. $2(x^2 + y^2) \le 10$, or $\underline{x^2 + y^2 \le 5}$.



We orient M using $\Omega = dx \wedge dy \wedge dz$.

Now, $d\omega = d(2x\, dy \wedge dz - z\, dx \wedge dy) = 2\, dx \wedge dy \wedge dz - dz \wedge dx \wedge dy =$
$= dx \wedge dy \wedge dz$. Therefore

$$\int_M d\omega = \iint_{x^2+y^2 \le 5} \left(\int_{x^2+y^2-6}^{4-x^2-y^2} 1\, dz \right) dx \wedge dy = \iint_{x^2+y^2\le 5} \left(4 - x^2 - y^2 - (x^2+y^2-6)\right) dx\, dy =$$
$$= \int_0^{2\pi} \int_0^{\sqrt{5}} (10 - 2r^2)\, r\, dr\, d\theta = 2\pi \int_0^{\sqrt{5}} (10r - 2r^3)\, dr = 2\pi \left(\frac{10r^2}{2} - \frac{r^4}{2} \right) \Big|_0^{\sqrt{5}} =$$
$$= 2\pi \left(25 - \frac{25}{2} \right) = 25\pi.$$

The boundary ∂M is made up of two pieces. We parameterize them as graphs: $T_1(u, v) = (u, v, 4 - u^2 - v^2)$
$T_2(u, v) = (u, v, u^2 + v^2 - 6)$.

Note that T_2 gives its piece of ∂M the wrong orientation: we need downward pointing normal, while graph parameterization gives upward normal. Therefore

$$\int_{\partial M} \omega = \int_{u^2+v^2\le 5} T_1^*\omega - \int_{u^2+v^2\le 5} T_2^*\omega = \int_{u^2+v^2\le 5} \left(2u\, dv \wedge d(4-u^2-v^2) - \right.$$

$$\left. -(4-u^2-v^2)\, du \wedge dv \right) - \int_{u^2+v^2\le 5} 2u\, dv \wedge d(u^2+v^2-6) - (u^2+v^2-6)\, du \wedge dv =$$

$$= \int \left(2u\, dv \wedge (-2u\, du) - (4-u^2-v^2)\, du \wedge dv \right) - \int \left(2u\, dv \wedge (2u\, du) - (u^2+v^2-6)\, du \wedge dv \right)$$

$$= \int_{u^2+v^2\le 5} \left(4u^2 + (u^2+v^2-4) + 4u^2 + (u^2+v^2-6) \right) du \wedge dv$$

$$= \int_{u^2+v^2\le 5} (8u^2 + 2(u^2+v^2) - 10)\, du\, dv = \int_0^{2\pi} \int_0^{\sqrt5} (8r^2\cos^2\theta + 2r^2 - 10)\, r\, dr\, d\theta =$$

$$= \int_0^{2\pi} \int_0^{\sqrt5} (8r^3\cos^2\theta + 2r^3 - 10r)\, dr\, d\theta = \int_0^{2\pi} \left(2r^4\cos^2\theta + \frac{r^4}{2} - 5r^2 \right) \Big|_0^{\sqrt5} d\theta =$$

$$= \int_0^{2\pi} \left(\frac{25}{2} - 25 + 25(\cos 2\theta + 1) \right) d\theta = 25\pi + \sin 2\theta \Big|_0^{2\pi} = 25\pi.$$

#12(a) $\frac{1}{3} \int_{\partial M} x\, dy \wedge dz - y\, dx \wedge dz + z\, dx \wedge dy =$ (Stokes)

$$= \int_M \frac{1}{3} d(x\, dy \wedge dz - y\, dx \wedge dz + z\, dx \wedge dy) =$$

$$= \int_M \frac{1}{3}(dx \wedge dy \wedge dz - dy \wedge dx \wedge dz + dz \wedge dx \wedge dy) = \int_M \frac{1}{3} 3\, dx \wedge dy \wedge dz$$

$=$ volume of M.

(b) This is similar. One only needs to observe that

$$d(x_1\, dx_2 \wedge \dots \wedge dx_n - x_2\, dx_1 \wedge dx_3 \wedge \dots \wedge dx_n + \dots)$$
$$= n\, dx_1 \wedge dx_2 \wedge \dots \wedge dx_n$$

8.4

#16 $d(x_1 x_3\, dx_2 \wedge dx_4) = x_3\, dx_1 \wedge dx_2 \wedge dx_4 + x_1\, dx_3 \wedge dx_2 \wedge dx_4$

$\neq x_2\, dx_2 \wedge dx_3 \wedge dx_4$. False

#18 $d(x_1 x_2\, dx_1 \wedge dx_2 + x_2 x_3\, dx_1 \wedge dx_3 + x_1 x_3\, dx_2 \wedge dx_3) =$

$= 0 + x_3\, dx_2 \wedge dx_1 \wedge dx_3 + x_3\, dx_1 \wedge dx_2 \wedge dx_3 = 0$

False.

#19 If ω is an n-form, then $d\omega$ is an $(n+1)$ form. But any $(n+1)$-form on $\mathbb{R}^n$ is zero. Therefore $d\omega = 0$.

True.