INVERSE FUNCTION THEOREM IMPLIES IMPLICIT FUNCTION THEOREM

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We show that the implicit function theorem can be deduced from the inverse function theorem. We consider only the case of $f : \mathbb{R}^2 \to \mathbb{R}$. The proof in higher dimensions is analogous. The only real difference is a more cumbersome notation.

We assume that there is a point $(a, b) \in \mathbb{R}^2$ so that $\frac{\partial f}{\partial y}(a, b) \neq 0$. We want to show that there is a neighborhood U of a in $\mathbb{R}$, V of b in $\mathbb{R}$ and a function $g : U \to V$ so that

$$f(x, g(x)) = c$$

for $x \in U$. Here, $c = f(a, b)$.

Proof. Consider the map

$$H(x, y) := (x, f(x, y)).$$

It is a map from $\mathbb{R}^2$ to $\mathbb{R}^2$. We will argue that it is invertible near (a, b). Indeed, its differential $DH(a, b)$ is

$$\begin{pmatrix} 1 & 0 \\ \frac{\partial f}{\partial x}(a, b) & \frac{\partial f}{\partial y}(a, b) \end{pmatrix}$$

Hence $\det DH(a, b) = 1 \cdot \frac{\partial f}{\partial y}(a, b)$, which is not zero by assumption. Hence $DH(a, b)$ is invertible. Hence, by the inverse function theorem, the map H is invertible near (a, b). Denote the inverse by G. It is of the form

$$G(u, v) = (G_1(u, v), G_2(u, v))$$

for some real-valued functions G_2, G_1 defined on a neighborhood of $H(a, b) = (a, f(a, b)) = (a, c)$ in $\mathbb{R}^2$. Since G and H are inverses of each other,

$$(u, v) = H(G(u, v)) = H(G_1(u, v), G_2(u, v)) = (G_1(u, v), f(G_1(u, v), G_2(u, v)))$$

for all (u, v) near (a, c). Therefore

$$u = G_1(u, v) \tag{1}$$

and

$$v = f(G_1(u, v), G_2(u, v)). \tag{2}$$

Plugging (1) into (2) we get:

$$v = f(u, G_2(u, v))$$

for all u near a, all v near c. Now let $v = c$, $u = x$. We get

$$c = f(x, G_2(x, c)).$$

Take

$$g(x) = G_2(x, c).$$

We are done. □

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