# Rigid body dynamics 1

Suppose we have a rigid body constrained to pivot freely about a fulcrum. We place the origin of our reference frame at the fulcrum.

We think of the body as a collection of $N$ ($\sim 10^{23}$) particles with positions $x_i \in \mathbb{R}^3$, $1 \le i \le N$, and masses $m_i$.

We assume that the distances $\|x_i - x_j\|$ are independent of time – the body is rigid. Let $(x_1^0, \ldots x_N^0)$ denote the set of positions of the particle at time 0. Then at time $t$ there is an orthogonal matrix $A(t) \in SO(3)$ so that

$$x_i(t) = A(t)\, x_i^0.$$

Hence the configuration space $Q$ of our system is $SO(3)$, which is a 3 dimensional manifold.

$$Q = \{ A \in M_{3\times 3}(\mathbb{R}) \mid AA^T = I \}$$

Note 1: By the regular value theorem, $\forall A \in SO(3)$ the tangent space $T_A SO(3)$ is given by

$$T_A SO(3) = \{ B \in M_{3\times 3}(\mathbb{R}) \mid BA^T + AB^T = 0 \}$$

Note 2 If the body evolves along a path $A(t)$, the kinetic energy of the $i^{th}$ particle at time $t$ is

$$\frac{m_i}{2} \langle \tfrac{d}{dt} A(t) x_i^0, \tfrac{d}{dt} A(t) x_i^0 \rangle$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^3$.

Hence the total kinetic energe $KE$ of the system is

$$KE = \sum_{i=1}^{N} \frac{m_i}{2} \langle \tfrac{d}{dt} A(t) x_i^0, \tfrac{d}{dt} A(t) x_i^0 \rangle$$

Define a Riemannian metric $g$ on $SO(3)$ as follows: at $A \in SO(3)$, $V_1, V_2 \in T_A SO(3)$

$$g_A(V_1, V_2) := \sum_{i=1}^{N} m_i \langle V_1 x_i^0, V_2 x_i^0 \rangle$$

With this definition of the metric, the kinetic energy of our rigid body moving along a trajectory $A(t) \in SO(3)$ (more pedantically $A : (a,b) \to SO(3)$ is a path in $SO(3)$) is given by 

$$KE = \frac{1}{2}\, g_{A(t)}(\dot{A}(t), \dot{A}(t)) \quad \left( = \frac{1}{2}\sum_{i=1}^{N} m_i \langle \dot{A}(t)x_i^o, \dot{A}(t)x_i^o\rangle \right)$$

<u>Definition</u> A metric $g$ on $SO(3)$ is <u>left-invariant</u> if $\forall B, A \in SO(3)$ $\forall\, V, W \in T_A SO(3)$

$$g_{BA}(BV, BW) = g_A(V,W).$$

<u>Note</u>: The definition implicitly uses the following

<u>Fact</u> If $V \in T_A SO(3)$ then $BV \in T_{BA} SO(3)$ $\forall B \in SO(3)$

<u>Exercise</u>: Prove the fact.

<u>Proposition</u> The kinetic energy metric on $SO(3)$ is left-invariant.

<u>Proof</u> For any two vectors $v_1, v_2 \in \mathbb{R}^3$ and for any $B \in SO(3)$

$$\langle Bv_1, Bv_2\rangle = \langle v_1, v_2\rangle.$$

Hence, $\forall\, V, W \in T_A SO(3)$

$$g_{BA}(BV, BW) = \sum_{i=1}^{N} m_i \langle BVx_i^o, BWx_i^o\rangle = \sum_{i=1}^{N} m_i \langle Vx_i^o, Wx_i^o\rangle$$
$$= g_A(V,W). \qquad \square$$

<u>Conclusion</u> If the only forces on the rigid body are constraint forces, the corresponding Lagrangian $L : TSO(3) \to \mathbb{R}$ is given by

$$(1) \qquad L(A, v) = \frac{1}{2} g_A(v,v)$$

where $g$ is a left-invariant metric on $SO(3)$.

We now add gravity: the potential energy of the $i^{th}$ particle is $m_i \times \text{height} \times \underbrace{\text{universal constant}}_{=\gamma} = \gamma m_i \langle x_i, \vec{e}_3\rangle$ where $\vec{e}_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$

$\Rightarrow$ The total potential energy $V(A)$ in the configuration $A \in SO(3)$

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is given by the formula

$$V(A) = \gamma \sum_{i=1}^{N} m_i \langle A x_i^0, \vec{e}_{gr}\rangle = \gamma \langle A(\sum m_i x_i^0), \vec{e}_{gr}\rangle$$

$$= \gamma m \langle A(\tfrac{1}{m} \sum m_i x_i^0), \vec{e}_{gr}\rangle$$

where $m = \sum m_i$.

<u>Note:</u> $e^0_{c.m.} = \frac{1}{m} \sum_{i=1}^{N} m_i x_i^0$ is the initial position of the center of mass of the body.

Therefore we can write (by setting $\gamma = 1$)

$$V(A) = m \langle A e^0_{cm}, \vec{e}_{gr}\rangle$$

<u>Conclusion:</u> In the presence of gravity the Lagrangian of the rigid body is of the form

$$(2) \quad L(A, v) = \tfrac{1}{2} g_A(v, v) + m \langle A \vec{e}^{\,0}_{c.m}, \vec{e}_{gr}\rangle.$$

We now investigate the symmetries of the Lagrangians (1)(2). We'd like to find the lagest Lie group $K$ acting on $TSO(3)$ which <u>preserves</u> the Lagrangians (1), (2). That is, for any $A \in SO(3)$, $\forall v \in T_A SO(3)$ $\forall a \in K$

$$L(a \cdot (A, v)) = L(A, v).$$

We have already seen that Lagrangian (1) is $SO(3)$ invariant for <u>any</u> rigid body. That is $SO(3) \subseteq K$.

But $K$ can be bigger if the body is symmetric.

Mathematically this is seen as follows.

(to be continued)