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Legendre transform, parth II.

6.1

Let M be a manifold, $L : TM \to \mathbb{R}$ a Lagrangian.
We assumed: for all $q \in M$, all $v \in T_qM$ the form

$$d^2L_{(q,v)} : T_qM \times T_qM \to \mathbb{R}$$

$$d^2L_{(q,v)}(v_1, v_2) = \frac{\partial^2}{\partial t_1 \partial t_2}\Big|_{(0,0)} L(q, v + t_1v_1 + t_2v_2)$$

is

positive definite.

We showed: under this assumption the image $\mathcal{O}$ of the Lendre transform

$$\mathcal{L}_L : TM \longrightarrow T^*M$$

is an open subset of T^*M and

$$\mathcal{L}_L : TM \longrightarrow \mathcal{O}$$

is a diffeomorphism. [Pretty soon we'll make a further assumption that $\mathcal{O} = T^*M$. This holds for the Lagrangians of the form $L(q,v) = \frac{1}{2} g_q(v,v) + V(v)$ where g is a Riemannian metric on M and $V : M \to \mathbb{R}$ is a smooth function, a potential.]

Def We define the Hamiltonian $H : \mathcal{O} \to \mathbb{R}$ associated with $L : TM \to \mathbb{R}$ by

$$H(q,p) = \langle p, \mathcal{L}_L^{-1}(q,p)\rangle - L(\mathcal{L}_L^{-1}(q,p)).$$

for all $q \in M$, $p \in T_q^*M$.
Here $\langle \cdot, \cdot \rangle : T_q^*M \times T_qM \to \mathbb{R}$ is the canonical pairing.

Remark In coordinates $\mathcal{L}_L(q_1 \dots q_n, v_1 \dots v_n) = (q_1 \dots q_n, \frac{\partial L}{\partial v_1}, \dots \frac{\partial L}{\partial v_n})$.
Hence

$$(H \circ \mathcal{L}_L)(q_1 \dots q_n, v_1 \dots v_n) = \sum_{i=1} \frac{\partial L}{\partial v_i} \cdot v_i - L(q_1 \dots q_n, v_1 \dots v_n)$$

Recall that there is a vector field X_L on TM associated to $L: TM \to \mathbb{R}$. The integral curves of X_L satisfy the Euler-Lagrange equations. Recall that we have arrived at these equations by looking at the extremal curves of the <u>action</u>

$$A_L(\gamma) = \int_a^b L(\gamma, \dot{\gamma})\,dt.$$

On the other hand we have a globally defined tautological 1-form $\alpha \in \Omega^1(T^*M)$, which in coordinates $(q_1 \ldots q_n, p_1 \ldots p_n) : T^*U \to \mathbb{R}^n \times \mathbb{R}^n$ is given by

$$\alpha = \sum p_i\, dq_i.$$

Then

$$\omega = d\alpha$$

is a globally defined 2-form on T^*M. In coordinates

$$\omega = \sum dp_i \wedge dq_i.$$

Hence ω is <u>nondegenerate</u>.

Consequently $H: \mathcal{O} \to \mathbb{R}$ and ω together define a Hamiltonian vector field

$$X_H = \sum_{i=1}^{?} \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i}.$$

The coordinate-free definition of X_H is

$$\omega(X_H, \cdot) = -dH$$

(Alternatively we could have set $\omega = -d\alpha$ and defined X_H by

$$\omega(X_H, \cdot) = dH.$$

Both conventions are used.)

Our goal is to prove:

Theorem 1 Let $L: TM \to \mathbb{R}$ be a Lagrangian so that the corresponding Legendre transform

$$\mathcal{L}_L : TM \to T^*M$$

is a diffeomorphism. Then $\forall q \in M$, $\forall v \in T_q M$

$$(*) \quad (D\mathcal{L}_L)_{(q,v)} X_L(q,v) = X_H(\mathcal{L}_L(q,v))$$

To prove the theorem, we need some differential geometry. We start with:

① Definition Let M, N be two manifolds. A continuous map $F: M \to N$ is smooth (equivalently C^∞) if it is smooth in coordinates: $\forall x \in M$ $\exists$ coordinates chart $\varphi: \mathcal{U} \to \mathbb{R}^m$ (with $x \in \mathcal{U}$) a coordinate chart $\psi: V \to \mathbb{R}^n$ (with $f(x) \in V$) so that $\psi \circ F \circ \varphi^{-1}: \varphi^{-1}(\mathcal{U} \cap F^{-1}(V)) \to \psi(V)$ is C^∞.

Special case: $N = \mathbb{R}$. Then $\psi: \mathbb{R} \to \mathbb{R}$, $\psi(y) = y$ is a coordinate chart on N. Thus $f: M \to \mathbb{R}$ is C^∞ if $\forall x \in M$, $\forall$ coordinate chart $\varphi: \mathcal{U} \to \mathbb{R}^m$,

$$f \circ \varphi^{-1}: \varphi(\mathcal{U}) \to \mathbb{R}$$

is C^∞.

Exercise if $F: M \to N$ is C^∞ and $f: N \to \mathbb{R}$ is C^∞ then so is

$$f \circ F: M \to \mathbb{R}.$$

② Differentials If $F: M \to N$ is a smooth map between two manifolds, we define its differential

$$DF_x : T_x M \to T_{F(x)} N \quad \text{by}$$
$$(DF_x(v))\, f = v(f \circ F)$$
for all $f \in C^\infty(N)$, all $v \in T_x M$.

Exercise Check that DF_x is linear.

③ Vector fields on manifolds

Def A vector field X on a manifold M assigns to each point $x \in M$ a vector $X(x) \in T_x M$ so that the map
$$X : M \to TM, \quad x \mapsto X(x)$$
is C^∞.

Remark The smoothness of X amounts to: for any coordinate chart $\varphi = (q_1 \ldots q_n) : U \to \mathbb{R}^n$ on M
$$X(x) = \sum X_i(x) \frac{\partial}{\partial q_i}\Big|_x$$
where $X_1 \ldots X_n : U \to \mathbb{R}$ are functions.
X is smooth $\Leftrightarrow$ $X_1, \ldots, X_n$ are all smooth.

④ Differential forms on manifolds

Definition: A differential 1-form α on a manifold M assigns to each point $x \in M$ a covector $\alpha_x \in T_x^* M$ so that the map
$$\alpha : M \to T^* M$$
is smooth.

Notation $\Omega^1(M)$ = space of differential 1-forms on M.

and $X: Q \to TQ$ is a vector field the contraction $\iota(X)\omega \in \Omega^1(Q)$ is a 1-form defined by

$$(\iota(X)\omega)_x(v) = \omega_x(X(x), v)$$

for all $x \in Q$, all $v \in T_x Q$.

It will be useful to look at the linear algebra version of the contraction:

Let V be a vector space, $\omega \in Alt^2(V)$ a skew symmetric bilinear map. Then $\forall v \in V$ we have a covector $\iota(v)\omega \in V^*$. It is defined by

$$(\iota(v)\omega)(u) = \omega(v, u)$$

for all $u \in V$.

Exercise: The map

$$\iota: V \times Alt^2(V) \to V^*, \quad (v, \omega) \mapsto \iota(v)\omega$$

is bilinear: $\forall a_1, b_1, a_2, b_2 \in \mathbb{R}$, $v_1, v_2 \in V$, $\omega_1, \omega_2 \in Alt^2(V)$

$$\iota(a_1 v_1 + a_2 v_2)(b_1\omega_1 + b_2\omega_2) = \sum_{i,j=1}^{2} a_i b_j \, \iota(v_i)\omega_j.$$

Remark More generally, for any $k \geq 1$ we have a contraction ..

$$\iota: V \times Alt^k(V) \to Alt^{k-1}(V)$$

$$(\iota(v)\sigma)(u_1 \dots u_{k-1}) = \sigma(v, u_1 \dots u_{k-1})$$

$\forall v \in V$, $\sigma \in Alt^k(V)$, $u_1 \dots u_{k-1} \in V$.

Exercise: $\forall l_1, l_2 \in V^*$ $\forall v \in V$

$$\iota(v)(l_1 \wedge l_2) = l_1(v)\, l_2 - l_2(v)\, l_1$$

Remark The smoothess of $\alpha : M \to T^*M$ amounts to:

(i) In any coordinates $(q_1 \ldots q_n) : U \to \mathbb{R}^n$ on M

$$\alpha_x = \sum_{i=1}^n \alpha_i(x) dq_i$$

where $\alpha_1 \ldots \alpha_n : U \to \mathbb{R}$ are smooth functions.

(ii) For any smooth vector field X on M the function

$$\alpha(X) : M \to \mathbb{R}, \ x \mapsto \alpha_x(X(x))$$

is C^∞.

Definition A differential 2-form ω on a manifold M assigns to each point $x \in M$ an alternating bilinear form $\omega_x \in Alt^2(T_x M)$ so that

(1) $\forall$ smooth vector fields X, Y on M the function

$$\omega(X,Y) : M \to \mathbb{R}, \quad x \mapsto \omega_x(X(x), Y(x))$$

is C^∞.

One can show that (1) is equivalent to:

(2) for any coordinate chart $(q_1 \ldots q_n) : U \to \mathbb{R}^n$ on M

$$\omega = \sum_{i,j} \omega_{ij} \, dq_i \wedge dq_j$$

and $\omega_{ij} : U \to \mathbb{R}$ $1 \leq i,j \leq n$ are C^∞.

5 Exterior derivative d.

For a smooth function $f : M \to \mathbb{R}$ on a manifold M the 1-form $df \in \Omega^1(M)$ is defined by

$$df_x(v) = v(f)$$

for all $x \in M$, $\forall v \in T_x M$.

In coordinates

$$df = \sum \frac{\partial f}{\partial q_i} dq_i$$

For a 1-form $\alpha \in \Omega^1(M)$, $d\alpha$ is a 2-form.
In coordinates $d\alpha$ is defined by

$$d\left(\sum \alpha_i \, dq_i\right) = \sum_i d\alpha_i \wedge dq_i$$

For a 2-form $\omega \in \Omega^2(M)$, $d\omega$ is a 3-form.
In coordinates $d\omega$ is defined by

$$d\left(\sum \omega_{ij} \, dq_i \wedge dq_j\right) = \sum d\omega_{ij} \wedge dq_i \wedge dq_j$$

and so on.

In general d of a k-form is a $(k+1)$-form.

Def A k-form η is <u>closed</u> if $d\eta = 0$.
It is <u>exact</u> if there is a $(k-1)$ form ξ so that $\eta = d\xi$.

<u>Theorem</u> $d(d\xi) = 0$.
Hence any exact form is closed.
(we'll won't prove this).

"<u>Example</u>" The canonical 2-form ω on the cotangent T^*Q is exact by definition, hence is closed.
Alternatively, in coordinates

$$\omega = \sum dp_i \wedge dq_i$$

$$\Rightarrow d\omega = 0.$$

<u>Contractions</u> of vector fields and differential 2-forms.

If $\omega \in \Omega^2(Q)$ is a 2-form on a manifold M

<u>Exercise</u> $\forall\, l_1, \dots l_k \in V^*$, $\forall v \in V$

$$\iota(v)(l_1 \wedge \dots \wedge l_k) = \sum_{i=1}^{k} (-1)^{k-1} l_i(v)\, l_1 \wedge \dots \wedge \hat{l_i} \wedge \dots \wedge l_k$$

where

$l_1 \wedge \dots \wedge \hat{l_i} \wedge \dots \wedge l_k$ means that l_i was omitted.

(e.g. $l_1 \wedge \hat{l_2} \wedge l_3 := l_1 \wedge l_3$

$\hat{l_1} \wedge l_2 \wedge l_3 = l_2 \wedge l_3$ etc.)

Proof of theorem 1 on p. 6.3

By definition X_H is the unique vector field on T^*M so that

$$\iota(X_H)\, \omega = -dH.$$

Therefore, in order to prove that

$$(D\mathcal{L}_L)_{(q,v)}(X_L(q,v)) = X_H(\mathcal{L}_L(q,v)) \qquad \forall q \in M, v \in T_qM$$

it is enough to show that

$$(1) \quad \iota\big((D\mathcal{L}_L)_{(q,v)}(X_L(q,v))\big)\, \omega_{(q,p)} = -dH_{(q,p)}$$

where $(q,p) = \mathcal{L}_L(q,v)$. Since

$$(D\mathcal{L}_L)_{(q,v)} : T_{(q,v)} TM \to T_{(q,p)} T^*M$$

is an isomorphism, (1) is equivalent to

$$(2) \quad \big(\iota((D\mathcal{L}_L)_{(q,v)} X_L(q,v))\, \omega_{(q,p)}\big)\big((D\mathcal{L}_L)_{(q,v)} w\big) = -dH_{(q,p)}\big((D\mathcal{L}_L)_{(q,v)} w\big)$$

for all $w \in T_{(q,v)}(TM)$.

One more piece of differential geometry that we need:

<u>pullback</u>

Let $F : M \to N$ be a smooth map. For a k-form σ on N we <u>define</u> the k-form $F^*\sigma$, on M, the <u>pullback</u> of σ by F, by the equation

$$(F^*\sigma)_x (v_1, \dots v_k) = \sigma_{F(x)}(DF_x(v_1), \dots, DF_x(v_k))$$

for all $x \in M$, $v_1 \dots v_k \in T_xM$.

<u>Theorem 2</u> (1) For all smooth functions $f : N \to \mathbb{R}$ and all smooth maps $F : M \to N$

$$F^* df = d(f \circ F)$$

(2) If $\omega \in \Omega^2(N)$, $x_1 \ldots x_n : U \to \mathbb{R}^n$ coordinates on N, then

$$F^*\left(\Sigma' \omega_{ij}\, dx_i \wedge dx_j\right) = \Sigma\, (\omega_{ij} \circ F) \cdot d(x_i \circ F) \wedge d(x_j \circ F)$$

where $\omega = \Sigma\, \omega_{ij}\, dx_i \wedge dx_j$.

<u>Proof</u> omitted.

We now look at the RHS of (2) above.

$$-\,dH_{(q,p)}\left((D\mathcal{L}_L)_{(q,v)}\, w\right) = -\left(\mathcal{L}_L^* dH\right)_{(q,v)}(w) \underset{\uparrow}{=} -d\left(\mathcal{L}_L^* H\right)_{(q,v)}(w).$$

(Theorem 2 (1) above)

On the other hand, the LHS of (2) is

$$\omega_{(q,p)}\left((D\mathcal{L}_L)_{(q,v)}\, X_L(q,v), (D\mathcal{L}_L)_{(q,v)}\, w\right) = \left(\mathcal{L}_L^*\omega\right)_{(q,v)}\left(X_L(q,v), w\right)$$

$$= \left(\imath(X_L)\, \mathcal{L}_L^* \omega\right)_{(q,v)}(w)$$

Therefore, since $w \in T_{(q,v)}(TM)$ is arbitrary, it suffices to show:

(3). $-\,\imath(X_L)\, \mathcal{L}_L^* \omega = d(\mathcal{L}_L^* H)$

We next compute in coordinates. Choose coordinates $(q_1 \ldots q_n) : U \to \mathbb{R}^n$ on M. Let

$(q_1 \ldots q_n, v_1 \ldots v_n) : TU \to \mathbb{R}^n \times \mathbb{R}^n$ and

$(q_1 \ldots q_n, p_1 \ldots p_n) : T^*U \to \mathbb{R}^n \times \mathbb{R}^n$

denote the corresponding coordinates on TM and T^*M, respectively. In these coordinates

$$\mathcal{L}_L(q_1 \ldots q_n, v_1 \ldots v_n) = \left(q_1 \ldots q_n, \frac{\partial L}{\partial v_1}, \ldots \frac{\partial L}{\partial v_n}\right)$$

and

$$\omega = \Sigma\, dp_i \wedge dq_i.$$

Consequently $d(\mathcal{L}_L^* H) = d\left(\langle \mathcal{L}(q,v), v\rangle - L(q,v)\right) =$

$$= d\left(\sum_i \frac{\partial L}{\partial v_i} \cdot v_i - L(q,v)\right) = \sum_{i=1}^{n}\left(\cancel{\frac{\partial L}{\partial v_i} dv_i} + v_i\, d\left(\frac{\partial L}{\partial v_i}\right) - \frac{\partial L}{\partial q_i} dq_i - \cancel{\frac{\partial L}{\partial v_i} dv_i}\right)$$

$$= \sum_{i=1}^{n}\left(v_i\, d\left(\frac{\partial L}{\partial v_i}\right) - \frac{\partial L}{\partial q_i} dq_i\right)$$

Recall that $\quad X_L = \sum_k' \left(v_k \frac{\partial}{\partial q_k} + B_k \frac{\partial}{\partial v_k} \right)$

where

$$B_k = \sum_i \left(\left(\frac{\partial^2 L}{\partial v_\alpha \partial v_\beta} \right)^{-1} \right)_{ki} \left(\frac{\partial L}{\partial q_i} - \sum_j' \frac{\partial^2 L}{\partial q_j \partial v_i} \cdot v_j \right)$$

$-\mathcal{L}_L^* \omega = \sum dq_i \wedge d\left(\frac{\partial L}{\partial v_i} \right)$ by Theorem 2 (2), the formula for $\mathcal{L}_L$ in coordinates and the fact that $\omega = \sum dp_i \wedge dq_i$.

Finally

$$-\imath(X_L)\, \mathcal{L}_L^* \omega \overset{\text{Exercise 1 on p 6.7}}{=} \sum_{i,k} \left(\imath\left(v_k \frac{\partial}{\partial q_k} + B_k \frac{\partial}{\partial v_k} \right) dq_i \wedge d\left(\frac{\partial L}{\partial v_i} \right) \right) =$$

$$= \sum_{i,k} \Bigg(v_k \cdot \Big(\underbrace{dq_i \left(\tfrac{\partial}{\partial q_k} \right)}_{\delta_{ik}} \cdot d\left(\tfrac{\partial L}{\partial v_i} \right) - \underbrace{d\left(\tfrac{\partial L}{\partial v_i} \right) \left(\tfrac{\partial}{\partial q_k} \right)}_{\frac{\partial}{\partial q_k}\left(\frac{\partial L}{\partial v_i} \right)} \cdot dq_i \Big) +$$

$$+ B_k \Big(\underbrace{dq_i \left(\tfrac{\partial}{\partial v_k} \right)}_{=0} \cdot d\left(\tfrac{\partial L}{\partial v_i} \right) - d\left(\tfrac{\partial L}{\partial v_i} \right) \left(\tfrac{\partial}{\partial v_k} \right) dq_i \Big) \Bigg) =$$

$$= \sum_{i,k} \left(v_k \delta_{ik}\, d\left(\frac{\partial L}{\partial v_i} \right) - v_k \frac{\partial^2 L}{\partial q_k \partial v_i} dq_i - B_k \cdot \frac{\partial^2 L}{\partial v_k \partial v_i} dq_i \right) =$$

$$= \sum_i v_i\, d\left(\frac{\partial L}{\partial v_i} \right) - \cancel{\sum_{i,k} v_k \frac{\partial^2 L}{\partial q_k \partial v_i} dq_i} - \sum_i \left(\frac{\partial L}{\partial q_i} - \cancel{\sum_j \frac{\partial^2 L}{\partial q_j \partial v_i} v_j} \right) dq_i$$

(where we used $\sum_{k,i} B_k \frac{\partial^2 L}{\partial v_k \partial v_i} = \sum_{i,s} \underbrace{\left(\sum_k \left(\left(\frac{\partial^2 L}{\partial v_\alpha \partial v_\beta} \right)^{-1} \right)_{ks} \cdot \frac{\partial^2 L}{\partial v_k \partial v_i} \right)}_{\delta_{is}} \left(\frac{\partial L}{\partial q_s} - \sum_j \frac{\partial^2 L}{\partial q_j \partial v_s} v_j \right)$

$$= \sum_i \left(\frac{\partial L}{\partial q_i} - \sum_j \frac{\partial^2 L}{\partial q_j \partial v_i} v_j \right) \Bigg)$$

$$= \sum_i' \left(v_i\, d\left(\frac{\partial L}{\partial v_i} \right) - \frac{\partial L}{\partial q_i} dq_i \right) = d(\mathcal{L}_L^* H).$$

$\square$