What is geometrical mechanics?

It's a geometric way to think about conservative classical mechanical systems (with time-independent holonomic constraints)

Examples come from celestial mechanics, rigid body dynamics.

It is useful for quantization.

It's a great framework for Noether's theorem: symmetries give rise to conservation laws.

It is not so good for
- dissipative systems
- time dependent constraints
- open systems in general

(nonholonomic constraints can be handled, but this requires more work)

<u>A bit</u> of historical perspective.

Newton's law of motion, $F = ma$, or, more generally for n particles, a system of ODEs

$$\begin{cases} \frac{dp_1}{dt} = F_1(q_1 \ldots q_n, p_1 \ldots p_n) \\ \vdots \\ \frac{dp_n}{dt} = F_n(q_1 \ldots q_1, p_1 \ldots p_n) \end{cases}$$

$q_1 \ldots q_n \in \mathbb{R}^3$, $p_1 \ldots p_n \in \mathbb{R}^3$ is over 400 years old.

In principle, any any classical system can be analysed in terms of forces.
In practice this only works for the simplest systems.
For example, try explaining why an upright spinning top is stable when it's spinning fast enough and unstable when it is spinning slowly.

Better approaches - Lagrangian (variational) and Hamiltonian

The two approaches are equivalent for conservative systems with holonomic constraints.

Both approaches tell you how to write down vector fields on the appropriate phase spaces. Actual trajectories of physical systems are <u>integral curves</u> of these vector fields.

<u>Aside</u> If $U \subseteq \mathbb{R}^n$ is a region, a vector field X on U is a map $X = (X_1 \ldots X_n) : U \to \mathbb{R}^n$
$\gamma(t) = (\gamma_1(t), \ldots \gamma_n(t))$ is an <u>integral curve</u> of X if $\forall t$

$$\frac{d\gamma}{dt} = X(\gamma(t))$$

Equivalently
$$\begin{cases} \frac{d\gamma_1}{dt} = X_1(\gamma_1(t), \ldots \gamma_n(t)) \\ \frac{d\gamma_2}{dt} = X_2(\gamma_1(t), \ldots \gamma_n(t)) \\ \vdots \\ \frac{d\gamma_n}{dt} = X_n(\gamma_1(t), \ldots \gamma_n(t)) \end{cases}$$

Both Lagrangian and Hamiltonian approaches are geometric.

For Hamiltonian systems the geometry is given by
(1) a phase space (which is a manifold)
together with
(2) a symplectic form, i.e., a closed nondegenerate differential 2-form
[these terms will be defined later] and
(3) a Hamiltonian.
This is just a function. Physically it is the total energy of the system.

To understand differential 2-forms and to understand how a function + a symplectic form define a vector field, we need linear algebra.

The key concepts are (1) dual vector spaces and (2) skew-symmetric bilinear maps (also called "forms")
(3) nondegeneracy

We start with a real finite dimensional vector space V. By definition the dual vector space V^* is the space of linear functionals on V:

$$V^* = \{ \ell : V \to \mathbb{R} \mid \ell \text{ is linear} \}$$

Recall A linear map $T: V \to W$ between two finite dimensional vector spaces is completely determined by what it does to a basis.

Fix a basis $\{v_1 - v_n\}$ of V. Then, for any i, $1 \le i \le n$, there is a unique linear funtional $\eta_i : V \to \mathbb{R}$ with

$$\eta_i(v_j) = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

<u>Proposition</u> Let $\{\eta_1 - \eta_n\} \in V^*$ be as above. Then $\{\eta_1 - \eta_n\}$ is a <u>basis</u> of V^*.

<u>Proof</u> exercise

<u>Consequences</u> (1) $\dim V^* = \dim V$

(2) if $T : V \to V^*$ is linear and $\ker T = \{\vec{0}\}$, Then T is 1-1 and onto, hence invertible

<u>Exercise:</u> prove (2)

<u>Hint:</u> Rank/nullity theorem is useful.

<u>Def</u> Let V be a finite dimensional real vector space. A map/function

$$b : V \times V \longrightarrow \mathbb{R}$$

is <u>bilinear</u> if it is linear in each argument. That is, $\forall\, u, v_1, v_2 \in V$, $a_1, a_2 \in \mathbb{R}$

$$b(u, a_1 v_1 + a_2 v_2) = a_1\, b(u, v_1) + a_2\, b(u, v_2).$$

and

$$b(a_1 v_1 + a_2 v_2, u) = a_1\, b(v_1, u) + a_2\, b(v_2, u).$$

<u>Examples</u> (1) An inner product $(\cdot,\cdot)$ on a vector space V is bilinear

(2) if $l_1, l_2 \in V^*$ then

$$b(v_1, v_2) := l_1(v_1) \cdot l_2(v_2) \quad \text{is bilinear.}$$

<u>Proposition</u> The space $\mathrm{Bilin}(V,\mathbb{R})$ of bilinear maps on a vector space V is a real vector space: if $b_1, b_2 : V\times V \to \mathbb{R}$ are two bilinear maps, $a_1, a_2 \in \mathbb{R}$ then
$a_1 b_1 + a_2 b_2$
defined by
$$(a_1 b_1 + a_2 b_2)(v_1, v_2) = a_1\, b_1(v_1,v_2) + a_2\, b_2(v_1,v_2)$$
for all $v_1, v_2 \in V$
is bilinear.

<u>Proof</u> exercise.

<u>Example</u> $\forall\, \ell_1, \ell_2 \in V^*$
$$(\ell_1 \wedge \ell_2)(v_1,v_2) := \ell_1(v_1)\cdot \ell_2(v_2) - \ell_1(v_2)\, \ell_2(v_1)$$
is bilinear

<u>Definition</u> $b \in \mathrm{Bilin}(V,\mathbb{R})$ is <u>skew-symmetric</u> (a.k.a. <u>alternating</u>) if
$$b(v_1, v_2) = -b(v_2, v_1) \qquad \forall\, v_1, v_2 \in V$$

<u>Ex</u> $\forall\, \ell_1, \ell_2 \in V^*$, $\ell_1 \wedge \ell_2$ in example above is alternating

<u>Ex</u> $V = \mathbb{R}^n$, $A \in M_{n,n}(\mathbb{R})$ $n\times n$ skew-symmetric matrix: $A^T = -A$.
Then
$$\omega_A(v,w) := v^T A w$$
is alternating.

<u>Notation</u> (fairly standard)

$$Alt^2(V) = \{ \omega : V \times V \to \mathbb{R} \mid \omega \text{ bilinear and alternating} \}$$

<u>Definition</u> $\omega \in Alt^2(V)$ is <u>nondegenerate</u> if $\forall v \in V$

$$\omega(v, w) = 0 \ \forall w \implies v = 0.$$

<u>Ex</u> if $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ then $\omega_A(v, w) = v^T A w$ is nondegenerate.

(check this!)

"Better" way to think about <u>nondegeneracy</u>.

If $\omega \in Alt^2(V)$, then $\forall v \in V$

$$\omega(v, \cdot) : V \to \mathbb{R}, \quad w \mapsto \omega(v, w)$$

is <u>linear</u>, ie. $\omega(v, \cdot) \in V^*$.

This gives us a map

$$\omega^\# : V \to V^* \qquad \omega^\#(v) = \omega(v, \cdot)$$

Since ω is bilinear, $\omega^\#$ is <u>linear</u> : $\forall a_1, a_2 \in \mathbb{R}$ $v_1, v_2 \in V$

$$\omega^\#(a_1 v_1 + a_2 v_2) = \omega(a_1 v_1 + a_2 v_2, \cdot) =$$
$$a_1 \omega(v_1, \cdot) + a_2 \omega(v_2, \cdot)$$
$$= a_1 \omega^\#(v_1) + a_2 \omega^\#(v_2)$$

$$\ker \omega^\# = \{ v \in V \mid \omega^\#(v) = 0 \}$$
$$= \{ v \in V \mid \omega^\#(v) w = 0 \ \forall w \in V \}$$
$$= \{ v \in V \mid \omega(v, w) = 0 \ \forall w \}$$

Thus:

ω is nondegernerate $\iff \ker \omega^\# = \{0\}$.

$\iff \omega^\# : V \to V^*$ is injective

$\Leftrightarrow \omega^{\#}: V \to V^*$ is an isomorphism (see p 4)

Conclusion If $\omega \in \mathrm{Alt}^2(V)$ is nondegenerate, then

$$\omega^{\#}: V \longrightarrow V^*, \quad v \mapsto \omega(v, \cdot)$$

is invertible. Thus, $\forall\, \ell \in V^*$ $\exists!$ $v = v(\ell)$ s.t.

$$\omega(v(\ell), w) = \ell(w) \qquad \forall\, w \in V.$$

Or, the equation in v

$$\omega(v, \cdot) = \ell$$

has a unique solution for all $\ell \in V^*$.

Exercise $V = \mathbb{R}^2$ $\omega(v, w) = v^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} w$

$\ell = (a \;\; b) \in (\mathbb{R}^2)^*$

Solve

$$\omega(v, \cdot) = \ell.$$

Hint $(v_1 \; v_2)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \;?$

Now we are ready to deal with fields of vectors, covectors and alternating bilinear forms.

0. A field X of vectors on a region $U \subseteq \mathbb{R}^n$ is a vector field.

We can think of it as an n-tuple of real valued functions

$$X = (X_1 \ldots X_n): U \to \mathbb{R}^n.$$

Recall a vector v at a point $q \in U$ defines a directional derivative D_v by

$$(D_v f)(q) = \frac{d}{dt}\Big|_0 f(q + tv).$$

We have a correspondence:

vectors at q $\longleftrightarrow$ directional derivatives at q.

Under this correspondence the standard basis vector $\vec{e}_i = (0, \dots, \underset{i^{th}\text{ slot}}{1}, \dots 0)^T$ corresponds to $\frac{\partial}{\partial x_i}\Big|_q$.

$T_q U$ = space of tangent vectors at q

$= \{ \sum a_i \frac{\partial}{\partial x_i}\big|_q \mid a_1 \dots a_n \in \mathbb{R} \}$.

Under this identification

$$X(q) = (X_1(q) \dots X_n(q)) \longleftrightarrow \sum X_i(q) \frac{\partial}{\partial x_i}\Big|_q .$$

(Thus $\{ \frac{\partial}{\partial x_1}\big|_q, \dots \frac{\partial}{\partial x_n}\big|_q \}$ is a basis of $T_q U$)

1. A field α of covectors on a region $U \subseteq \mathbb{R}^n$ is called a 1-form.

$\forall q \in U$, $T_q^* U := (T_q U)^*$, the cotangent space at q

The basis of $T_q^* U$ dual to $\{ \frac{\partial}{\partial x_1}\big|_q, \dots \frac{\partial}{\partial x_n}\big|_q \}$ is denoted by

$$(dx_1)_q, \dots (dx_n)_q$$

By <u>definition</u>

$$(dx_i)_q \left(\frac{\partial}{\partial x_j}\Big|_q \right) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j . \end{cases}$$

A 1-form α on U is thus an expression

$$\alpha = \sum \alpha_i \, dx_i , \quad \alpha_1, \dots \alpha_n : U \to \mathbb{R}$$

smooth functions.

2. Every smooth function $f: U \to \mathbb{R}$ defines a 1-form df. There are several ways to think about it.

(i) (almost formal) $df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$

(ii) $\forall q \in \mathcal{U}$, df_q should be a linear functional
$$df_q : T_q\mathcal{U} \to \mathbb{R}$$
we define it by
$$df_q(v) = \frac{d}{dt}\Big|_0 f(q+tv) \overset{\text{chain rule}}{=} \sum \frac{\partial f}{\partial x_i}(q)\, v_i$$
$$= \sum \frac{\partial f}{\partial x_i}(q) (dx_i)_q(v)$$

This again gives us
$$df = \sum \frac{\partial f}{\partial x_i} dx_i .$$

---

3. A field of alternating bilinear forms on $\mathcal{U} \subseteq \mathbb{R}^n$ is called a differential 2-form.

Thus ω is a 2-form $\iff$
$$\omega = \sum_{i<j} \omega_{ij}\, dx_i \wedge dx_j$$
where ω_{ij} are smooth functions.

Thus, $\forall q \in \mathcal{U}$, $v, w \in T_q\mathcal{U}$
$$\omega_q(v, w) = \Big(\sum_{i<j} \omega_{ij}(q)\, (dx_i)_q \wedge (dx_j)_q\Big)(v, w)$$
$$= \sum_{i<j} \omega_{ij}(q) \cdot (v_i w_j - v_j w_i)$$

Note that this secretly uses the follow theorem:

__Thm__ Let V be a vector space with a basis $\{v_1 \dots v_n\}$.
Let $\{\eta_1, \dots \eta_n\}$ be the dual basis of V^*.
Then $\{\eta_i \wedge \eta_j \mid i<j\}$ is a basis of $\mathrm{Alt}^2(V)$.

__Exercise:__ prove the Theorem.

Hints (i) Given $\omega \in Alt^2(V)$ show that it is uniquely determined by $\binom{n}{2}$ numbers $\{\omega(v_i, v_j) \mid i<j\}$. #1

(ii) Use (i) to show that

$$\omega = \sum_{i<j} \omega(v_i, v_j)\, \eta_i \wedge \eta_j$$

Notation $\Omega^2(U)$ = space of differential 2-forms on a region $U \subseteq \mathbb{R}^n$.

Def $\omega \in \Omega^2(U)$ is nondegenerate if $\forall q \in U$ $\omega_q \in Alt^2(T_qU)$ is nondegenerate.

Q. What's so great about nondegenerate 2-forms?

A. A nondegenerate 2-form ω and a function $H: U \to \mathbb{R}$ define a vector field X_H by

$$\omega_q(X_H(q), \cdot) = dH_q.$$

"Example" $U = \mathbb{R}^2$ with coordinates q and p. $\omega = dq \wedge dp$ is nondegenerate.

Let

$$H(q,p) = \frac{1}{2m} p^2 + V(q) \quad \text{where } m>0,$$

$V(q)$ a function.

We claim that

(1) $$X_H(q,p) = \frac{p}{m}\frac{\partial}{\partial q} - V'(q)\frac{\partial}{\partial p}$$

(2) an integral curve $(q(t), p(t))$ of X_H solves

$$(*) \begin{cases} \dfrac{dq}{dt} = \dfrac{1}{m} p \\ \dfrac{dp}{dt} = -V'(q) \end{cases}$$

Indeed, $\forall\, v \in \mathbb{R}^2$

$$(dq \wedge dp)\left(\frac{\partial}{\partial q}, v\right) = \underbrace{dq\left(\frac{\partial}{\partial q}\right)}_{=1} \cdot dp(v) - dq(v)\, \underbrace{dp\left(\frac{\partial}{\partial q}\right)}_{=0}$$

$$= dp(v)$$

$$\Rightarrow \quad \omega^{\#}\left(\frac{\partial}{\partial q}\right) = dp.$$

Similarly, $\omega^{\#}\left(\frac{\partial}{\partial p}\right) = -dq$

$$\Rightarrow \quad \omega^{\flat}(dp) = \frac{\partial}{\partial q}, \qquad \omega^{\flat}(dq) = -\frac{\partial}{\partial p}.$$

$$\Rightarrow \omega^{\flat}(dH) = \omega^{\flat}\left(\frac{p}{m}\, dp + V'(q)\, dq\right)$$

$$= \frac{p}{m} \frac{\partial}{\partial q} - V'(q) \frac{\partial}{\partial q}$$

<u>Note</u> (★) on p10 gives

$$m \frac{d^2 q}{dt^2} = m \cdot \frac{d}{dt}\left(\frac{dq}{dt}\right) = m \frac{d}{dt}\left(\frac{1}{m} p\right) = \frac{dp}{dt} = -V'(q)$$

This is Newton's law of motion for a particle on a line subject to the conservative force with the potential $V(q)$.