Mechanics notes 2: pullback of differential forms.

1. Assumed background from vector calculus.

Let $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$ be open sets and $F = (F_1 \ldots F_m) : U \to V$ a differentiable function. Then for any $q \in U$ we have the matrix of partials

$$DF(q) := \left(\frac{\partial F_i}{\partial x_j}(q) \right) = \begin{pmatrix} \frac{\partial F_1}{\partial x_1}(q) & \frac{\partial F_1}{\partial x_2}(q) & \cdots & \frac{\partial F_1}{\partial x_n}(q) \\ \frac{\partial F_2}{\partial x_1}(q) & \cdots & \cdots & \vdots \\ \vdots & & & \end{pmatrix}$$

We identify this matrix with a linear map

$$DF(q) : T_q U = \mathbb{R}^n \longrightarrow \mathbb{R}^m = T_{F(q)} V$$

$$DF(q) v = \left(\frac{\partial F_i}{\partial x_j}(q) \right) \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \qquad \forall v \in T_q U$$

Aside It is often better to think of $DF(q) : \mathbb{R}^n \to \mathbb{R}^m$ as a linear approximation to $F(q + \Delta q) - F(q)$. One proves that

$$\lim_{\Delta q \to \vec{0}} \frac{1}{\|\Delta q\|} \left(F(q + \Delta q) - F(q) - DF(q) \Delta q \right) = \vec{0}$$

Alternatively we can define F to be differentiable at q if there is a linear map $A : \mathbb{R}^n \to \mathbb{R}^m$ so that

$$\lim_{\Delta q \to \vec{0}} \frac{1}{\|\Delta q\|} \left(F(q + \Delta q) - F(q) - A \Delta q \right) = \vec{0}$$

It is not very hard to prove that if such A exists, it is unique and that $A = \left(\frac{\partial F_i}{\partial x_j}(q) \right)$.

[Subtle point: existence of partials does not guarantee differentiability; existence of continuous partials $\frac{\partial F_i}{\partial x_j}$ does.]

Important basic result about derivatives: <u>chain rule</u>.

If $F: U \to V$ is differentiable at q and $G: V \to W$ is differentiable at $F(q)$ then $G \circ F$ is differentiable at q as well and

$$D(G \circ F)(q) = DG(F(q)) \circ DF(q)$$

composition of linear maps.

Two special cases of DF: $f: U \to \mathbb{R}$ and $\gamma: (a,b) \to U$.

1. Suppose $f: U \to \mathbb{R}$ is a smooth function. Then $\forall q \in U$ we have

$$Df(q): T_q U \to T_{f(q)} \mathbb{R} \simeq \mathbb{R}.$$

We therefore can think of $Df(q)$ as a linear functional

$$\boxed{Df(q) = df_q : T_q U \to \mathbb{R}}$$

df_q is sometimes called the <u>total differential of f</u>. at q

As we vary q, we get a differential 1-form df.

[Sometimes, incorrectly, df is called the gradient of f. But the gradient ∇f is a vector field; and it depends not just on f but also on a Riemannian metric. We'll talk about the distinction between 1-forms and vector fields later on.]

2. Let $\gamma = (\gamma_1, \ldots, \gamma_n): (a,b) \to U \subseteq \mathbb{R}^n$ be a curve.

Then $\forall t \in (a,b)$, $D\gamma(t): T_t(a,b) = \mathbb{R} \to T_{\gamma(t)} U = \mathbb{R}^n$,

$$D\gamma(t) = \begin{pmatrix} \dot{\gamma}_1(t) \\ \vdots \\ \dot{\gamma}_n(t) \end{pmatrix}, \quad \dot{\gamma}_i(t) := \frac{d\gamma_i}{dt}(t)$$

Note that $1 \in T_t(a,b)$ and $D\gamma(t)1$ is a <u>vector</u>.

We set $\boxed{\dot{\gamma}(t) = D\gamma(t)1 \in T_{\gamma(t)} U}$

$\dot{\gamma}(t)$ is the tangent vector at $\gamma(t)$ to the curve γ; the velocity of γ at t.

For $\gamma : (a,b) \to \mathcal{U}$ and $f : \mathcal{U} \to \mathbb{R}$ chain rule implies that

$$\frac{d}{dt} f(\gamma(t)) = df_{\gamma(t)}(\dot{\gamma}(t))$$

Similarly, given $v \in T_q \mathcal{U}$ $\exists$ a curve $\gamma(t)$ with $\gamma(0) = q$, $\dot{\gamma}(0) = v$. Then for any differentiable map

$$F : \mathcal{U} \to V \subseteq \mathbb{R}^m,$$

$$DF(q)v = DF(q)\dot{\gamma}(0) = DF(q)\, d\gamma(t)1 = \frac{d}{dt}\Big|_0 F(\gamma(t)).$$

Pull back

If $\mathcal{U} \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$ are open and $F : \mathcal{U} \to V$ smooth, then any differential k form $\omega \in \Omega^k(V)$ can be pulled back by F to define $F^*\omega \in \Omega^k(V)$. We'll do it for $k = 0$ (functions), 1 and 2.

Pull back is defined in two steps.

Step 1 "Tangent vectors push forward."

Given a smooth map $F : \mathcal{U} \to V$ as above, $q \in \mathcal{U}$, $v \in T_q \mathcal{U}$ we have

$$DF(q)v \in T_{F(q)} V.$$

Since $v = \dot{\gamma}(0)$ for a curve $\gamma : (a,b) \to \mathcal{U}$. with $\gamma(0) = q$

(*) $$DF(q)v = \frac{d}{dt}\Big|_0 F(\gamma(t))$$

by the chain rule; see above.

Aside

Later on we will think of configuration spaces of physical systems as manifolds.

For example, for a planar pendulum the configuration space is S^1. For a double planar pendulum it is $S^1 \times S^1$. For a spherical pendulum the configuration space is S^2. For a spinning top the configuration space is $SO(3)$.

Recall that the special orthogonal group $SO(3)$ is defined by

$$SO(3) = \{3\times 3 \text{ matrix } A \mid A^T A = I,\ \det A = 1\}.$$

For a rigid body the configuration space is the (special) Euclidean group

$$SEuc(3) = \left\{ \left(\begin{array}{c|c} A & b \\ \hline 0\,0\,0 & 1 \end{array}\right) \;\middle|\; A \in SO(3),\ b \in \mathbb{R}^3 \right\}$$

We'll need to define tangent spaces of manifolds and differentials of maps between manifolds. We'll see that DF_q is defined in such a way that (*) still holds.

Step 2

Pull back of 0-forms, i.e. functions:

if $\varphi : V \to \mathbb{R}$ is smooth, $F: U \to V$ is smooth

$$F^*\varphi := \varphi \circ F \in C^\infty(U)$$

In other words

$$(F^*\varphi)(q) = \varphi(F(q)) \quad \forall q \in U,\ \forall \varphi \in C^\infty(V).$$

Now let $\alpha \in \Omega^1(V)$ be a 1-form.

For $v \in T_q U$, $DF_q(v) \in T_{F(q)} V$. $\alpha_{F(q)}$ is a linear map from $T_{F(q)} V$ to $\mathbb{R}$. So it makes sense to define

(**) $$(F^*\alpha)_q (v) := \alpha_{F(q)}(DF(q)v) \quad \forall q \in U\ \forall v \in T_q U$$

One can use (**) to compute $F^*\alpha$, but there is a better way.

Note first that we can multiply 1-forms by functions (0-forms).

$\forall\, \alpha \in \Omega^1(V)$, $\forall\, \varphi \in C^\infty(V) = \Omega^0(V)$

$$(\varphi\alpha)_x(w) := \varphi(x)\,\alpha_x(w) \qquad \forall x \in V,\ \forall w \in T_x V$$

<u>Claim</u>: $\forall\, \alpha \in \Omega^1(V)$, $\forall\, \varphi \in C^\infty(V)$ $\forall\, F : U \to V$

$$F^*(\varphi\cdot\alpha) = (F^*\varphi)\cdot F^*\alpha$$

<u>Proof</u> $\forall\, q \in U$, $\forall\, v \in T_q U$

$$F^*(\varphi\cdot\alpha)_q(v) = (\varphi\alpha)_{F(q)}(DF_q(v))$$

$$= \varphi(F(q))\,\alpha_{F(q)}(DF_q(v))$$

$$= (F^*\varphi)(q)\cdot(F^*\alpha)_q(v)$$

<u>Claim 2</u> $F^* : \Omega^1(V) \to \Omega^1(U)$ preserves addition:

$\forall\, \alpha, \beta \in \Omega^1(V)$

$$F^*(\alpha+\beta) = F^*\alpha + F^*\beta.$$

<u>Proof</u> exercise.

Ask me if you get stuck.

<u>Claim 3</u> Suppose $f \in C^\infty(V)$ Then

$$F^*(df) = d(f\circ F)\ (= d(F^*f)).$$

<u>Proof</u> This is chain rule in disguise. For $q \in V$, $v \in T_q U$

$$F^*(df)_q(v) = df_{F(q)}(DF(q)v) = (Df(F(q))\circ DF(q))(v)$$

$$= D(f\circ F)(q)(v)$$

$$= d(f\circ F)_q(v)$$

(see $\S$ 2.2 for $Df(q) = df_q$)

Putting claims 1–3 together we get

<u>Proposition</u> Let $F : U \to V$ be smooth and $\alpha = \sum_{j=1}^m \alpha_j(y)\,dy_j \in \Omega^1(V)$

Then $F^*\alpha = \sum_{i=1}^m \alpha_i(F(x))\,dF_i$ where $F = (F_1, \ldots F_m)$

Postponing a proof for a moment, we compute an example.

$\underline{Ex}$ $F(r,\theta) = (r\cos\theta, r\sin\theta)$,

$$\alpha = \frac{1}{x^2+y^2}(x\,dy - y\,dx) \in \Omega^1(\mathbb{R}^2 \setminus \{0\})$$

$$F^*\alpha = \frac{1}{r^2}\big(r\cos\theta\, d(r\sin\theta) - r\sin\theta\, d(r\cos\theta)\big)$$

$$= \frac{1}{r^2}\Big[r\cos\theta\Big(\frac{\partial}{\partial r}(r\sin\theta)dr + \frac{\partial}{\partial\theta}(r\sin\theta)d\theta\Big) -$$

$$- r\sin\theta\cdot\Big(\frac{\partial}{\partial r}(r\cos\theta)dr + \frac{\partial}{\partial\theta}(r\cos\theta)d\theta\Big)\Big]$$

$$= \frac{1}{r^2}\big(\cancel{r\cos\theta\cdot\sin\theta\,dr} + r\cos\theta\, r\cos\theta\, d\theta -$$

$$- \cancel{r\sin\theta\cdot\cos\theta\,dr} + r\sin\theta\, r\sin\theta\, d\theta\big)$$

$$= \cancel{\frac{1}{r^2}}\,\cancel{(r^2\cos^2\theta + r^2\sin^2\theta)}\, d\theta = d\theta.$$

Proof of proposition. Note that $F^*y_i = y_i \circ F = F_i$. Hence

$$(F^*\alpha)_x = F^*\Big(\sum \alpha_i\, dy_i\Big)_x \overset{\text{claim 2}}{=} \sum F^*(\alpha_i\, dy_i)_x$$

$$\overset{\text{claim 2}}{=} \sum (F^*\alpha_i)_x\,(F^*dy_i)_x$$

$$\overset{\text{claim 3}}{=} \sum \alpha_i(F(x))\, d(F^*y_i)_x = \sum \alpha_i(F(x))\,(dF_i)_x$$

What about 2-forms?

Given $\omega \in \Omega^2(V)$ and $F: U \to V$ we define $F^*\omega \in \Omega^2(U)$ by

$$(F^*\omega)_q(v,w) := \omega_{F(q)}\big(DF(q)v, DF(q)w\big)$$

for all $q \in U$, $v, w \in T_q U$.

Proposition Given $\omega = \sum_{i<j} \omega_{ij}\, dy_i \wedge dy_j \in \Omega^2(V)$
and $F = (F_1 \ldots F_m) : U \to V$

$$(F^*\omega)_x = \sum \omega_{ij}(F(x))\, dF_i \wedge dF_j .$$

Example $F(r,\theta) = (r\cos\theta,\, r\sin\theta)$
$\omega = dy_1 \wedge dy_2 \in \Omega^2(\mathbb{R}^2)$

To compute the example we also need to "recall" that

(1) $dx_i \wedge dx_j = - dx_j \wedge dx_i$
(hence $dx_i \wedge dx_i = - dx_i \wedge dx_i \Rightarrow dx_i \wedge dx_i = 0$)

(2) $f\, dx_i \wedge g\, dx_j = fg\, dx_i \wedge dx_j$ $\forall$ functions f, g.

We now compute:

$$F^*(dy_1 \wedge dy_2) = d(r\cos\theta) \wedge d(r\sin\theta)$$
$$= (\cos\theta\, dr - r\sin\theta\, d\theta) \wedge (\sin\theta\, dr + r\cos\theta\, d\theta)$$
$$= \cos\theta\, dr \wedge \sin\theta\, dr + \cos\theta\, dr \wedge r\cos\theta\, d\theta$$
$$- r\sin\theta\, d\theta \wedge \sin\theta\, dr - r\sin\theta\, d\theta \wedge r\cos\theta\, d\theta$$
$$= \cos\theta\sin\theta\, \underbrace{dr \wedge dr}_{=0} + r\cos^2\theta\, dr \wedge d\theta - r\sin^2\theta\, d\theta \wedge dr$$
$$- r^2 \sin^2\theta\, \underbrace{d\theta \wedge d\theta}_{=0}$$
$$= (r\cos^2\theta + r\sin^2\theta)\, dr \wedge d\theta$$
$$= r\, dr \wedge d\theta .$$

The only new ingredient we need to prove the proposition is

Claim 4 $\forall\, \alpha, \beta \in \Omega^1(V)$
$$F^*(\alpha \wedge \beta) = F^*\alpha \wedge F^*\beta .$$

Proof $\forall q \in \mathcal{U}$, $\forall v, w \in T_q \mathcal{U}$

$$F^*(\alpha \wedge \beta)_q (v, w) = (\alpha \wedge \beta)_{F(q)} (DF(q)v, DF(q)w)$$

$$= \alpha_{F(q)}(DF(q)v) \cdot \beta_{F(q)}(DF(q)w) -$$

$$- \alpha_{F(q)}(DF(q)w) \cdot \beta_{F(q)}(DF(q)v)$$

$$= (F^*\alpha)_q(v) \cdot (F^*\beta)_q(w) - (F^*\alpha)_q(w) \cdot (F^*\beta)_q(v)$$

$$= (F^*\alpha)_q \wedge (F^*\beta)_q \ (v, w) \equiv (F^*\alpha \wedge F^*\beta)_q (v, w).$$