Mechanics notes 3: Orthogonal group is a manifold

Recall

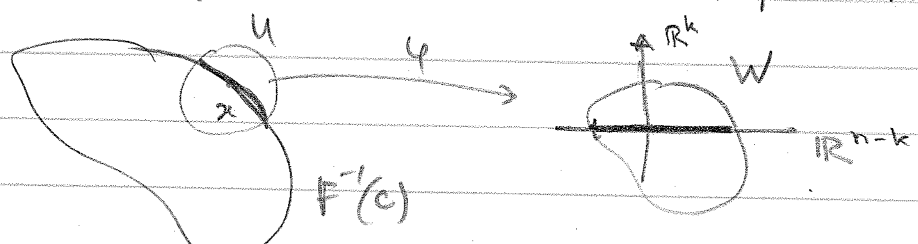
Definition A point $c \in \mathbb{R}^k$ is a regular value of a smooth map $F: \mathbb{R}^n \to \mathbb{R}^k$ if $\forall x \in \mathbb{R}^n$ with $F(x) = c$,

$$DF(x): T_x \mathbb{R}^n \to T_c \mathbb{R}^k$$

is onto.

Theorem If $c \in \mathbb{R}^k$ is a regular value of $F: \mathbb{R}^n \to \mathbb{R}^k$ then $F^{-1}(c) := \{x \in \mathbb{R}^n \mid F(x) = c\}$ is a manifold. Moreover, $\forall x \in F^{-1}(c)$ there is a local change of coordinates on $\mathbb{R}^n$, $\varphi: U \to W$ ($U, W \subseteq \mathbb{R}^n$ open subsets, φ diffeomorphism) so that

$$\varphi(F^{-1}(c) \cap U) = \{y \in W \mid y_{n-k+1} = y_{n-k+2} = \cdots = y_n = 0\}$$



(See Math 423 for the case where $n = 3$, $k = 1$ and $F^{-1}(c)$ is a surface).

Application $O(n) = \{A \text{ } n \times n \text{ matrix} \mid A^T A = I\}$ is a manifold.

Proof Consider $F: M_n(\mathbb{R}) = \text{space of } n\times n \text{ matrices} \to Sym(n)$

(where $Sym(n)$ = $n \times n$ symmetric matrices $= \{X \in M_n(\mathbb{R}) \mid X = X^T\}$)

$$F(X) = X^T X$$

Then $O(n) = F^{-1}(I)$.

We argue that I is a regular value of F.

<u>Note</u> $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$, $\mathrm{Sym}(n) \cong \mathbb{R}^{n+\binom{n}{2}} = \mathbb{R}^{\frac{n(n+1)}{2}}$

We first compute $DF(I)$: $\forall X \in M_n(\mathbb{R})$

$$DF(I)X = \frac{d}{dt}\Big|_0 F(I+tX) = \frac{d}{dt}\Big|_0 (I+tX)^T(I+tX)$$

$$= \frac{d}{dt}\Big|_0 \left(I + tX^T + tX + t^2X^TX\right) = X + X^T$$

Since $\forall Y \in \mathrm{Sym}(n)$, $Y = \frac{1}{2}(Y+Y^T) = DF(I)\left(\frac{1}{2}Y\right)$,

$DF(I): T_I M_n(\mathbb{R}) \to T_I \mathrm{Sym}(n)$ is onto.

<u>Remains to prove:</u> $\forall A \in O(n)$, $DF(A): T_A M_n(\mathbb{R}) \to T_I \mathrm{Sym}(n)$ is onto.

<u>Trick</u> : (i) $\forall A \in M_n(\mathbb{R})$ the map $L_A: M_n(\mathbb{R}) \to M_n(\mathbb{R})$, $L_A(B) := AB$ is C^∞

(ii) If additionally A is invertible, then L_A is invertible:

Indeed $(L_{A^{-1}} \circ L_A)(B) = A^{-1}(AB) = IB = B$

and $(L_A \circ L_{A^{-1}})(B) = AA^{-1}B = B$ for all B.

(iii) It follows from the chain rule that for any invertible $n\times n$ matrix A, the differential

$$(DL_A)(B): T_B M_n(\mathbb{R}) \to T_{AB}(M_n(\mathbb{R}))$$

is invertible.

(iv) For any $A \in O(n)$, $\forall X \in M_n(\mathbb{R})$

$$F(L_A(X)) = F(AX) = (AX)^T AX = X^TA^TAX$$
$$= X^T I X = X^T X = F(X).$$

Chain rule now implies: $\forall A \in O(n)$

$$DF(I) = D(F\circ L_A)(I) = DF(A) \circ (DL_A)(I)$$

We have computed that $DF(I)$ is onto.

$$\Rightarrow \quad DF(A) = \underbrace{DF(I)}_{\text{onto}} \circ \underbrace{\left(DL_A(I)\right)^{-1}}_{\text{isomorphism}}$$

is onto as well.

$\Rightarrow$ I is a regular value of $F: M_n(\mathbb{R}) \to Sym(n)$.

$\Rightarrow$ $F^{-1}(I) = O(n)$ is a manifold. $\square$

__Note__ $\dim O(n) = \dim M_n(\mathbb{R}) - \dim Sym(n)$

$$= n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}.$$

One can show further that $O(n)$ is a __Lie group__. That is, the multiplication

$$m: O(n) \times O(n) \to O(n)$$

$$m(A, B) = AB \qquad \text{is } C^\infty.$$

Note a subtle point: it is clear that

$$O(n) \times O(n) \to M_n(\mathbb{R}), \quad (A, B) \mapsto AB$$

is C^∞. It is less clear that m is smooth as a map into the manifold $O(n)$.