Tangent spaces

If $\Sigma \subseteq \mathbb{R}^3$ is a surface, and $p \in \Sigma$ a point, we know how to define the tangent space $T_p\Sigma$ to Σ at p: it's the space of "infinitesimal displacements"

$$T_p\Sigma = \{\dot{\gamma}(0) \mid \gamma : I \to \Sigma \text{ a curve, } \gamma(0) = p\}.$$

There are two issues with the definition:

(i) It's not clear that $T_p\Sigma$ is a vector space

(ii) It's not obvious how to generalize this definition for an abstract manifold M.

In math 423 we got around (i) by picking a parametrization $F : U \to V \cap \Sigma$ ($U \subseteq \mathbb{R}^2$, $V \subseteq \mathbb{R}^3$ open, $p \in V \cap \Sigma$) and showing that

$$T_p\Sigma = \{DF(q)v \mid v \in \mathbb{R}^2\},$$

where $q = F^{-1}(p)$. That is, for any curve $\gamma : I \to \Sigma$ with $\gamma(0) = p$, there is a vector $v \in \mathbb{R}^2$ so that

$$\dot{\gamma}(0) = DF(q)v$$

This is not hard to generalize if our manifold M is sitting inside some $\mathbb{R}^n$. More formally, if M is <u>embedded</u> in $\mathbb{R}^n$. This means: $M \subset \mathbb{R}^n$ and $\forall p \in M$ $\exists$ an open set $V \subseteq \mathbb{R}^n$ containing p and a diffeomorphism $\varphi : V \to W \subseteq \mathbb{R}^n$ (W open) so that

$$\varphi(V \cap M) = W \cap (\mathbb{R}^\ell \times \{0\}) = \{x \in W \mid x_{\ell+1} = x_{\ell+2} = \dots = x_n = 0\}$$

(i.e. here M has dimension ℓ and $\varphi^{-1}|_{W \cap (\mathbb{R}^\ell \times \{0\})}$ will give a local parametrization of M)

We then set $q = \varphi(p)$ and define

$$T_pM = \{D\varphi^{-1}(q)v \mid v \in \mathbb{R}^\ell \times \{0\}\}$$

Lemma With the notation above,

$$T_pM = \{ \dot\gamma(0) \mid \gamma: I \to M,\ \gamma(0)=p \}$$

Proof Given $\gamma: I \to M$, let $\tau = \varphi\circ\gamma$ and $v = \dot\tau(0)$

Then $D(\varphi^{-1})(q)v = \frac{d}{dt}\big|_0 (\varphi^{-1}(\tau(t)) = \frac{d}{dt}\big|_0 (\varphi^{-1}\circ\varphi\circ\gamma)(t) = \dot\gamma(0).$ □

Why is this not completely satisfactory?

(1) In this set up T_pM is a subspace of $\mathbb{R}^n$ and it seems to depend on how M sits inside $\mathbb{R}^n$.

(2) What do we do when M is given abstractly, and not as a subspace of some $\mathbb{R}^n$

What do we do if M is given as a subset of two different $\mathbb{R}^n$'s ?

There are two equivalent solutions to the problem and one usually uses both:

(i) tangent vectors are equivalent classes of curves (they are "infinitesimal displacements")

(ii) tangent vectors are abstract "directional derivatives" called <u>derivations</u>.

Let M be a manifold, $p \in M$.

Def Two curves $\gamma, \tau: I \to M$, $\gamma(0) = \tau(0) = p$ are <u>tangent to each other</u> at p if for any $f \in C^\infty(M)$

$$\frac{d}{dt}\Big|_0 f(\gamma(t)) = \frac{d}{dt}\Big|_0 f(\tau(t)).$$

We write: $\gamma \sim_p \tau$

We'd like to think of γ and τ as representing the <u>same</u> infinitesimal displacement at p.

Formally $\sim_p$ is an equivalence relation on curves in M passing through p and we define a tangent vector v to M at p to be the equivalence class of such curves:

$$v = [\gamma].$$

Problem It's not at all clear that these "displacements" form a vector space.
However, for any smooth function f on M it makes sense to differentiate f in the direction of $v = [\gamma]$:

$$(*) \quad v(f) := \frac{d}{dt}\Big|_0 f(\gamma(t))$$

By definition of $\sim_p$ it doesn't matter which curve in v we use to compute the R.H.S. of $(*)$.

Note that $(*)$ defines a linear map $v: C^\infty(M) \to \mathbb{R}$
Moreover, $\forall f, g \in C^\infty(M)$ $\forall \gamma: I \to M$ with $\gamma(0) = p$

$$\frac{d}{dt}\Big|_0 (f \cdot g)(\gamma(t)) = \left(\frac{d}{dt}\Big|_0 f(\gamma(t))\right) g(\gamma(0)) + f(\gamma(0)) \frac{d}{dt}\Big|_0 g(\gamma(t))$$

$$= v(f)\, g(p) + f(p)\, v(g).$$

This is supposed to motivate:
Def A derivation at $p \in M$ is a linear map

$$v: C^\infty(M) \to \mathbb{R} \quad \text{so that}$$

$$v(fg) = v(f)\, g(p) + f(p)\, v(g)$$

$$T_pM = \{ v: C^\infty(M) \to \mathbb{R} \mid v \text{ is a derivation} \}.$$

Exercise $\forall a, b \in \mathbb{R}$ $\forall v, w \in T_pM$, $av + bw$ defined by

$$(av + bw)(f) = a\, v(f) + b\, w(f) \qquad \forall f \in C^\infty(M)$$

is a derivation.
Hence T_pM is a vector space.

<u>Theorem</u> Any derivation is an infinitesimal displacement along some curve. Thus the two definitions agree.

< the proof is not hard once we understand a relation between derivations and coordinates >

Let M be a manifold, $\varphi = (x_1 \ldots x_n) : U \to \mathbb{R}^n$ a coordinate chart. Then each x_i is a smooth function on U.

<u>Def</u> (of $\frac{\partial}{\partial x_i}\Big|_p \in T_pM$). For $f \in C^\infty(M)$ we set

$$\frac{\partial}{\partial x_i}\Big|_p f = \frac{d}{dt}\Big|_0 (f \circ \varphi^{-1})(\varphi(p) + t e_i)$$
$$= \frac{\partial}{\partial u_i}\Big|_{\varphi(p)} (f \circ \varphi^{-1}),$$

where $(u_1 \ldots u_n)$ are coordinates on $\mathbb{R}^n$.

<u>Exercise</u> $\frac{\partial}{\partial x_i}\Big|_p$ is a derivation.

<u>Theorem</u> (requires work) $\left\{ \frac{\partial}{\partial x_i}\Big|_p \right\}_{i=1}^n$ is a basis of T_pM. Hence $\dim(T_pM) = \dim M$.

<u>Aside</u> For $f \in C^\infty(M)$ and $v \in T_pM$, $v(f)$ only depends on the values of f near p, as it should. This is another theorem. It follows that if $U \subseteq M$ is an open set and $p \in U$, then

$$T_pU = T_pM.$$

(Yes, I know, this is a bit fussy)

<u>Consequence</u> If M is a manifold, $\mathcal{U} \subseteq M$ open, $f \in C^\infty(\mathcal{U})$ then $\forall p \in \mathcal{U}$ we have a <u>linear</u> map

$$df_p : T_pM \to \mathbb{R}$$

defined by

$$df_p(v) := v(f) \qquad (\text{i.e. } df_p \in T_p^*M := (T_pM)^*)$$

This is because $\forall\, v, w \in T_pM \quad \forall a, b \in \mathbb{R}$

$$df_p(av + bw) = (av + bw)(f) = a\,v(f) + b\,w(f)$$
$$= a\, df_p(v) + b\, df_p(w)$$

<u>Def</u> df_p is the <u>differential</u> of f at p.

Now back to coordinates $\varphi = (x_1 \ldots x_n) : \mathcal{U} \to \mathbb{R}^n$.
Since $x_1 \ldots x_n : \mathcal{U} \to \mathbb{R}$ are smooth, they define, $\forall p \in \mathcal{U}$, differentials $(dx_1)_p, \ldots, (dx_n)_p \in T_p^*M$.

<u>Lemma</u> $(dx_1)_p, \ldots (dx_n)_p$ is a basis of T_p^*M dual to the basis $\{\frac{\partial}{\partial x_1}|_p, \ldots \frac{\partial}{\partial x_n}|_p\}$.

<u>Proof</u> $(x_1 \circ \varphi^{-1}, \ldots x_n \circ \varphi^{-1})(u_1 \ldots u_n) = (x_1 \ldots x_n) \circ \varphi^{-1}(u_1 \ldots u_n)$

$$= (\varphi \circ \varphi^{-1})(u_1 \ldots u_n) = (u_1 \ldots u_n)$$

$$\Rightarrow (x_i \circ \varphi^{-1})(u_1 \ldots u_n) = u_i \qquad \forall u \in \varphi(\mathcal{U})$$

$$\Rightarrow (dx_j)_p\left(\frac{\partial}{\partial x_i}\Big|_p\right) = \frac{\partial}{\partial x_i}\Big|_p (x_j) = \frac{\partial}{\partial u_i}\Big|_{\varphi(p)} (x_j \circ \varphi^{-1})$$

$$= \frac{\partial}{\partial u_i}\Big|_{\varphi(p)} (u_j) = \delta_{ij}. \qquad \square$$

We now recall the tricks with bases and dual bases:
If V is a vector space with a basis $\{v_1 \ldots v_n\}$ and $\{l_1 \ldots l_n\}$ is the dual basis of V^*, then

$\forall\, w \in V$ (*) $w = \sum_{i=1}^{n} l_i(w)\, v_i = \sum \langle l_i | w \rangle\, v_i$

$\forall\, \eta \in V^*$

$$\eta = \sum \eta(v_i)\, l_i = \sum \langle \eta | v_i \rangle\, l_i$$

Consequences (1) $\forall\ f \in C^\infty(U)$ $\forall p \in U$

$$df_p = \sum df_p\left(\frac{\partial}{\partial x_i}\Big|_p\right)(dx_i)_p = \sum \frac{\partial}{\partial x_i}\Big|_p(f)\ (dx_i)|_p$$

(2) If $\varphi = (x_1 \ldots x_n)$, $\psi = (y_1 \ldots y_n) : U \to \mathbb{R}^n$ are two coordinate charts then

$$\boxed{dy_j = \sum \frac{\partial}{\partial x_i}(y_j)\ dx_i}$$

Note $\frac{\partial}{\partial x_i}(y_j) = \frac{\partial}{\partial u_i}(y_j \circ \varphi^{-1}) = \frac{\partial F_j}{\partial u_i}$

where

$$F = (F_1 \ldots F_n) = \psi \circ \varphi^{-1}.$$

Similarly

(3) $$\frac{\partial}{\partial y_j} = \sum dx_i\left(\frac{\partial}{\partial y_j}\right)\frac{\partial}{\partial x_i} = \sum_{i=1}^{n}\left(\frac{\partial x_i}{\partial y_j}\right)\frac{\partial}{\partial x_i}$$