Group actions and Noether's theorem 1.

Definition An action of a Lie group G on a manifold M is a smooth map $G \times M \to M$, $(g, m) \mapsto g \cdot m$ so that (1) The identity element $e \in G$ acts trivially:

$$e \cdot m = m \quad \forall m \in M$$

(2)

$$g_1 \cdot (g_2 \cdot m) = (g_1 g_2) \cdot m \qquad \forall g_1, g_2 \in G,\ \forall m \in M$$

$\uparrow$ multiplication in G

Examples 1. $SO(3)$ acts on $(\mathbb{R}^3)^n$ by

$$A \cdot (r_1, \ldots r_n) = (Ar_1, \ldots Ar_n)$$

2. $\mathbb{R}^3$ acts on $(\mathbb{R}^3)^n$ by translations

$$v \cdot (r_1, \ldots, r_n) = (r_1 + v, \ldots r_n + v)$$

3. $SO(3)$ acts on itself by left multiplication:

$$SO(3) \times SO(3) \to SO(3), \quad (A, B) \mapsto \underbrace{AB}_{\text{mult. in } SO(3)}$$

Exercise: check that the three examples above are examples of actions.

Non example: $SO(3)$ acts on itself by right multiplication

$$SO(3) \times SO(3) \to SO(3), \quad (A, B) \mapsto BA$$

It's not an action since

$$(A_1 A_2) \cdot B = B A_1 A_2,$$

$$A_1 \cdot (A_2 \cdot B) = (B A_2) A_1$$

and, for general A_1, A_2 we don't have

$$B A_1 A_2 = B A_2 A_1.$$

Theorem An action of a Lie group G on a manifold M gives rise to actions of G on TM and on T^*M. 10.2

Proof Since G acts on M, $\forall g \in G$ we have a C^∞ map

$$g_M : M \to M, \quad g_M(x) = g \cdot x \quad \forall x \in M$$

g_M is invertible, since

$$((g^{-1})_M \circ g_M)(x) = g^{-1} \cdot (g \cdot x) = (g^{-1}g) \cdot x = e \cdot x = x$$

and, similarly, $(g_M \circ (g^{-1})_M)(x) = x \quad \forall x \in M$.

Given $x \in M$, $g_M(x) = g \cdot x$.

$\Rightarrow (dg_M)_x$ sends $T_x M$ to $T_{g \cdot x} M$

This gives us a map

$$G \times TM \to TM, \quad g \cdot v = (dg_M)_x v$$

Aside: if $v = \dot\gamma(0)$, $\gamma : I \to M$ a curve with $\gamma(0) = x$

Then $g \cdot v = (dg_M)_x(\dot\gamma(0)) = \frac{d}{dt}\Big|_0 (g \cdot \gamma(t))$

Example $SO(3)$ acts on itself by left multiplication

$\gamma(t)$ a curve in $SO(3)$

$$\Rightarrow \forall A \in SO(3) \qquad A \bullet (\dot\gamma(0)) = \frac{d}{dt}\Big|_0 (A\,\gamma(t))$$

$$= A \cdot \dot\gamma(0)$$

multiplication in $M_3(\mathbb{R})$.

Thus, for $B \in SO(3)$, $v \in T_B SO(3)$

$$A \cdot (B, v) = (AB, Av) \in T_{AB}(SO(3))$$

(when we think of $TSO(3)$ as a subspace of $M_3(\mathbb{R}) \times M_3(\mathbb{R})$:

$$TSO(3) = \{(A, v) \in SO(3) \times M_3(\mathbb{R}) \mid vA^T + Av^T = 0\}$$

<u>Exercise</u> verify that the map

$$G \times TM \to TM$$

$$(g, (x, v)) \longmapsto (g.x, (dg_M)_x v)$$

for any $g \in G$, $x \in M$, $v \in T_x M$ <u>is</u> an action.

To get an action of G on T^*M, we proceed as follows: $\forall x \in M$, $\eta \in T_x^* M$, $\forall g \in G$

$(dg_M)_x : T_x M \to T_{g.x} M$ is an isomorphism whose inverse is

$$(d(g^{-1})_M)_{g.x} : T_{g.x} M \to T_x M.$$

Therefore, for any linear map $\eta : T_x M \to \mathbb{R}$

$$\eta \circ d((g^{-1})_M)_{g.x} =: d(g^{-1})_M{}^T \eta$$

is a linear map from $T_{g.x} M$ to $\mathbb{R}$.

Since $d((g^{-1})_M)_{g.x} : T_{g.x} M \to T_x M$ is an isomorph

$$d((g^{-1})_M)_{g.x}^T : T_x^* M \to T_{g.x}^* M$$

$$\eta \longmapsto \eta \circ d((g^{-1})_M)_{g.x}$$

is also an isomorphism.

Thus, $\forall g \in G$ we get a map

$$g_{T^*M} : T^*M \to T^*M$$

$$(x, \eta) \longmapsto (g.x, (d(g^{-1})_M)_{g.x}^T \eta)$$

<u>Exercise</u> Check that the map

$$G \times T^* M \to T^* M$$

$$(g, (x, \eta)) \longmapsto g_{T^*M}(x, \eta)$$

is an action of G on T^*M.

<u>Example</u> Consider the standard action of $SO(3)$ on $\mathbb{R}^3$: $(A, x) \longmapsto Ax$

$$A_{\mathbb{R}^3}(x) = Ax$$

$\forall x \in \mathbb{R}^3$, $\forall v \in \mathbb{R}^3$, $\gamma(t) = x + tv$ is a curve in $\mathbb{R}^3$ with $\gamma(0) = x$, $\dot{\gamma}(0) = v$

$$\Rightarrow (dA_{\mathbb{R}^3})_x(v) = \frac{d}{dt}\Big|_0 A(x+tv) = Av$$

$$\Rightarrow A_{T\mathbb{R}^3}(x, v) = (Ax, Av)$$

Similarly, $d(A^{-1}_{\mathbb{R}^3}) = A^{-1}$ ($= A^T$ since $A \in SO(3)$) and, if we identify $(\mathbb{R}^3)^*$ with <u>row</u> vectors, and $T^*\mathbb{R}^3 \simeq \mathbb{R}^3 \times (\mathbb{R}^3)^*$, then the induced action of $SO(3)$ on $T^*\mathbb{R}^3$ is given by

$$A \cdot (x, \eta) = (Ax, \eta A^T)$$

for all $x \in \mathbb{R}^3$, $\forall \eta \in T^*_x\mathbb{R}^3 \simeq (\mathbb{R}^3)^*$.

We are now in position to explain the main goal of the course:

$$\boxed{\text{Symmetries give rise to conservation laws}}$$

<u>Mathematical set-up</u>: We have a conservative classical system with the configuration space a manifold M, phase space T^*M and the dynamics is determined by a Hamiltonian

$$H : T^*M \to \mathbb{R}.$$

Recall how H defines a vector field on T^*M:

We have the tautological 1-form $\alpha \in \Omega^1(T^*M)$

which in coordinates is given by

$$\alpha = \sum p_i \, dq_i$$

The vector field X_H is defined by

$$\boxed{\iota(X_H)\, d\alpha = -dH}$$

Quick check: $dH = \sum \frac{\partial H}{\partial q_i} dq_i + \sum \frac{\partial H}{\partial p_i} dp_i$

Since $d\alpha = \sum dp_i \wedge dq_i$, X_H must be given by

$$X_H = \sum \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i}$$

Hence an integral curve $\gamma(t) = (q_1(t), \dots q_n(t), p_1(t) \dots p_n(t))$ must solve

$$\begin{cases} \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}(q_1(t) \dots p_n(t)) \\ \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}(q_1(t) \dots p_n(t)) \end{cases}$$

Now suppose we have an action of a Lie group G on M. This defines an action of G on T^*M.

Theorem Suppose $H \in C^\infty(T^*M)$ is G-invariant:

$\forall g \in G$, $x \in M$, $\eta \in T_x^* M$

$$H(x,\eta) = H(g \cdot (x,\eta))$$

Then for any vector $X \in T_e G = \mathfrak{g}$ there is a function $\mu^X : T^*M \to \mathbb{R}$ which is an integral of motion for H:

$\forall$ integral curve γ of X_H the function

$$t \longmapsto \mu^X(\gamma(t))$$

is constant

We now examine the mathematics that we need to prove the theorem, which is due to E. Noether.

1. <u>The exponential map.</u>

For any Lie group G there is a map
$$\exp : T_e G \longrightarrow G, \quad \text{the exponential map}$$
so that
(1) $\frac{d}{dt}\big|_0 (\exp tY) = Y$
(2) $\exp(t+s)Y = (\exp tY) \cdot \exp sY$ ← mult in G.

<u>Facts</u> If $G \subseteq GL(n, \mathbb{R})$
$$\exp(Y) = \sum_{n=0}^{\infty} \frac{1}{n!} Y$$

If $G = \mathbb{R}^n$ $\exp Y = Y$.

2. Vector fields induced by group actions:

If a group G acts on a manifold Q then $\forall Y \in \mathfrak{g} = T_eG$ we get a vector field Y_Q on Q.
$$Y_Q(q) = \frac{d}{dt}\Big|_0 (\exp tY) \cdot q$$

3. Suppose a Lie group G acts on a manifold M. Then G acts on T^*M and <u>this</u> action preserves the 1-form $\alpha \in \Omega^1(T^*M)$:
$$\boxed{(g_{T^*M})^* \alpha = \alpha}$$
for all $g \in G$.

(we'll prove this later)

4. <u>Lie derivatives</u>

Let X be a vector field on a manifold Q and let $\varphi_t : Q \to Q$ denote its flow. Recall: this means that for any $q \in Q$
$$\frac{d}{dt} \varphi_t(q) = X(\varphi_t(q)), \quad \forall t$$

The Lie derivative $L_X\sigma$ of a k-form $\sigma \in \Omega^k(Q)$ is defined by

$$(L_X\sigma)_x = \frac{d}{dt}\Big|_0 (\varphi_t^*\sigma)_x \qquad \forall x \in Q$$
$$= \lim_{t\to 0} \frac{1}{t}\left((\varphi_t^*\sigma)_x - \sigma_x\right)$$

Theorem (Cartan's magic formula)
$$L_X\sigma = d(\iota(X)\sigma) + \iota(X)d\sigma.$$

< I don't plan to prove this theorem... >

5. If a Lie group G acts on a manifold Q then $\forall Y \in \mathfrak{g}$, the flow $\varphi_t : Q \to Q$ of the induced vector field Y_Q is given by
$$\varphi_t(q) = (\exp tY)\cdot q$$

We now put (3), (4) and (5) together:
Suppose we start with an action of a Lie group G on a manifold M. Then $\forall Y \in \mathfrak{g}$
$$\left((\exp tY)_{T^*M}\right)^*\alpha = \alpha \qquad \forall t$$

$$\Rightarrow \quad 0 = \frac{d}{dt}\Big|_0 \left((\exp tY)_{T^*M}\right)^*\alpha = L_{Y_{T^*M}}\alpha =$$

$$= d\,\iota(Y_{T^*M})\alpha + \iota(Y_{T^*M})\,d\alpha$$
$$= d\left(\alpha(Y_{T^*M})\right) + \iota(Y_{T^*M})\,d\alpha$$

Set $\mu^Y = \alpha(Y_{T^*M})$

Then

$$(**) \qquad -d\mu^Y = \iota(Y_{T^*M})\,d\alpha$$

(**) says: The induced vector field Y_{T^*M} is Hamiltonian, and the corresponding Hamiltonian μ^Y is given by

$$\mu^Y(x,\eta) = \alpha_{(x,\eta)}(Y_{T^*M}(x,\eta)).$$

<u>Theorem</u> The functions $\mu^Y : T^*M \to \mathbb{R}$ are invariants of motion for any Hamiltonian vector field X_H defined by a G-<u>invariant</u> function $H : T^*M \to \mathbb{R}$.

<u>Proof</u> Since H is G-invariant, $\forall\, Y \in \mathfrak{g}$ we have

$$H(x,\eta) = H((\exp tY)\cdot(x,\eta))$$

for all $x \in M$, $\eta \in T^*_x M$, $t \in \mathbb{R}$.

$\Rightarrow$

$$0 = \frac{d}{dt}\Big|_0 H((\exp tY)\cdot(x,\eta)) = Y_{T^*M}(H)$$
$$= dH(Y_{T^*M})$$
$$= -d\alpha(X_H, Y_{T^*M}) \quad (\text{since } dH = -\iota(X_H)\, d\alpha)$$
$$= d\alpha(Y_{T^*M}, X_H) \quad (\text{since } d\alpha \text{ is skew-sym})$$
$$= (-d\mu^Y)(X_H)$$

$\Rightarrow$ $\forall$ integral curve $\gamma(t)$ of X_H

$$\frac{d}{dt}\mu^Y(\gamma(t)) = d\mu^Y(\dot\gamma(0)) = d\mu^Y(X_H(\gamma(0)))$$
$$= 0$$

$\therefore$ $t \mapsto \mu^Y(\gamma(t))$ is constant. $\square$

<u>Exercise:</u> Prove that H is a constant of motion for X_H.