# Group actions and Noether's theorem, 2.



We fill in the details left over from March 31 notes.

1. The tautological 1-form $\alpha_{T^*M} \in \Omega^1(T^*M)$.

Recall that in coordinates $(q_1 \ldots q_n) : M \supseteq U \to \mathbb{R}^n$ the corresponding coordinates $(q_1 \ldots q_n, p_1 \ldots p_n) : T^*U \to \mathbb{R}^n \times \mathbb{R}^n$ the tautological 1-form $\alpha = \alpha_{T^*M}$ is given by

$$\alpha = \sum p_i \, dq_i .$$

On the other hand, $\alpha$ has the following coordinate-free description. Suppose $x \in M$, $\eta \in T_x^*M$, $v \in T_\eta(T^*M)$. Then $d\pi_\eta : T_\eta(T^*M) \to T_{\pi(\eta)} M = T_x M$ where

$$\pi : T^*M \to M$$

is the canonical projection.

We set

$$\alpha_{(x,\eta)}(v) = \langle \eta, (d\pi)_\eta (v) \rangle$$

where $\langle \cdot, \cdot \rangle : T_x^*M \times T_x M \to \mathbb{R}$ is the canonical pairing.

**Lemma 1** The two definitions of $\alpha$ agree.

**Proof** Let $(q_1 \ldots q_n) : \underset{\substack{\cap \\ M}}{U} \to \mathbb{R}^n$, $(q_1 \ldots q_n, p_1 \ldots p_n) : \underset{\substack{\cap \\ T^*M}}{T^*U} \to \mathbb{R}^n \times \mathbb{R}^n$ be coordinate charts as above, let $x \in U$, $\eta \in T_x^* U = T_x^* M$, $v \in T_{(x,\eta)}(T^*M)$.

Then

① $\eta = \sum p_i(\eta) (dq_i)_x$ by definition of the induced coordinates $(q_1 \ldots q_n, p_1 \ldots p_n)$

② $v = \sum_{i=1}^{n} \left( v_i \frac{\partial}{\partial q_i}\Big|_{(x,\eta)} + v_{i+n} \frac{\partial}{\partial p_i}\Big|_{(x,\eta)} \right)$

for some $v_1 \ldots v_{2n} \in \mathbb{R}$.

<u>Claim</u> $d\pi_{(x,\eta)}\left(\frac{\partial}{\partial q_i}\Big|_{(x,\eta)}\right) = \frac{\partial}{\partial q_i}\Big|_x \quad \forall i$

$$d\pi_{(x,\eta)}\left(\frac{\partial}{\partial p_i}\Big|_{(x,\eta)}\right) = 0 \quad \forall i$$

<u>Exercise</u> prove the claim.

We now compute

$$\alpha_{(x,\eta)}(v) = \langle \eta, (d\pi)_{(x,\eta)}(v)\rangle = \langle \Sigma\, p_i(\eta)(dq_i)_x , \sum_j v_j \frac{\partial}{\partial q_j}\Big|_x\rangle$$

$$= \langle (\Sigma\, p_i\, dq_i)_{(x,\eta)}, \Sigma\, v_j \frac{\partial}{\partial q_j}\Big|_{(x,\eta)} + v_{j+n}\frac{\partial}{\partial p_j}\Big|_{(x,\eta)}\rangle$$

$$= (\Sigma\, p_i\, dq_i)_{(x,\eta)}(v). \qquad \square$$

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<u>Recall:</u> if $\varphi: M \to M$ is any diffeomorphism, we can lift it to a diffeomorphism $\tilde{\varphi}: T^*M \to T^*M$:

For $x \in M$, $\eta \in T^*_x M$

$$(*) \qquad \tilde{\varphi}(x,\eta) = (\varphi(x), (d\varphi^{-1})^T_{\varphi(x)}\,\eta)$$

<u>Note</u> By definition $(*)$ of $\tilde{\varphi}$

$$\pi \circ \tilde{\varphi} = \varphi \circ \pi$$

where $\pi: T^*M \to M$ is the canonical projection

<u>Consequence:</u> Let $G \times M \to M$ be an action of a Lie group $G$ on a manifold $M$, $G \times T^*M \to T^*M$ the corresponding lifted action (see p.10.3), $X \in \mathfrak{g} = \mathrm{Lie}(G)$ a vector and $X_M$, $X_{T^*M}$ the corresponding induced vector fields on $M$ and $T^*M$, respectively.

<u>Proposition</u>, $d\pi \circ X_{T^*M} = X_M \circ \pi$, where, as before,

$\pi : T^*M \to M$ is the canonical projection.

Proof By definition of a lifted action, for any $g \in G$, $x \in M$, $\eta \in T^*_x M$,

$$\pi(g\cdot(x,\eta)) = g\cdot x \quad (= g\cdot \pi(x,\eta))$$

$$\Rightarrow \quad d\pi(X_{T^*M}(x,\eta)) = \frac{d}{dt}\Big|_0 \pi((\exp tX)\cdot(x,\eta))$$

$$= \frac{d}{dt}\Big|_0 \exp tX \cdot \pi(x,\eta) \quad \text{by}$$

$$= X_M(\pi(x,\eta)) \quad (= X_M(x)) \qquad \square$$

<u>Proposition 2</u> For any diffeomorphism $\varphi : M \to M$

$$(\tilde{\varphi})^* \alpha_{T^*M} = \alpha_{T^*M},$$

where $\tilde{\varphi} : T^*M \to T^*M$ is the lift of $\varphi$ and $\alpha_{T^*M} = \alpha \in \Omega^1(T^*M)$ is the tautological 1-form.

Proof We use the fact that $\boxed{\oplus \; \pi \circ \tilde{\varphi} = \varphi \circ \pi.}$

$\forall x \in M$, $\eta \in T^*_x M$, $v \in T_{(x,\eta)}(T^*M)$

$$(\tilde{\varphi}^*\alpha)_{(x,\eta)}(v) = \alpha_{\tilde{\varphi}(x,\eta)}((d\tilde{\varphi})_{(x,\eta)}(v)) \quad \text{by def of } \tilde{\varphi}^*$$

$$= \langle (d\varphi^{-1})^T_{\varphi(x)}\eta, \; d\pi_{\tilde{\varphi}(x,\eta)} \circ (d\tilde{\varphi})_{(x,\eta)} v \rangle \quad \text{by def of } \alpha$$

$$= \langle \eta, \; (d\varphi^{-1})_{\varphi(x)} \circ d\pi_{\tilde{\varphi}(x,\eta)} \circ d\tilde{\varphi}_{(x,\eta)} v \rangle$$

$$= \langle \eta, \; \cancel{(d\varphi^{-1})_{\varphi(x)} \circ d\varphi_x} \circ d\pi_{(x,\eta)} v \rangle \quad \text{by } \oplus$$

$$= \langle \eta, \; d\pi_{(x,\eta)} v \rangle \quad \text{by chain rule (and the fact that } (\varphi^{-1}\circ\varphi)(x) = x)$$

$$= \alpha_{(x,\eta)}(v) \quad \text{by definition of } \alpha. \qquad \square$$

(Proposition 2 is fact #3 on p 10.6. We use it together with Cartan's formula to prove:

$\forall Y \in \mathfrak{g}$

$$\mu^Y : T^*M \to \mathbb{R}, \quad \mu^Y(x,\eta) = \alpha_{(x,\eta)}(Y_{T^*M}(x,\eta))$$

is a conserved quantity for any $G$-invariant Hamiltonian $H : T^*M \to \mathbb{R}$

Proposition 1 implies:

<u>Proposition 3</u> Let $G \times M \to M$, $G \times T^*M \to T^*M$ be the actions as in Proposition 1 and let

$$\mu^Y(x,\eta) = \alpha_{(x,\eta)}\left(Y_{T^*M}(x,\eta)\right)$$

Then $\mu^Y(x,\eta) = \langle \eta, Y_M(x) \rangle$.

<u>Proof</u> $\alpha_{(x,\eta)}(Y_{T^*M}(x,\eta)) = \langle \eta, d\pi_{(x,\eta)} Y_{T^*M}(x,\eta)\rangle$
$= \langle \eta, Y_M(x)\rangle$ by Prop 1.

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This concludes our proof of Noether's theorem (except for a proof of Cartan's formula).