Symmetries, conservation laws and reduction of the number of degrees of freedom

Our goal is to understand (a version of) a Theorem due to Marsden, Weinstein and Meyer, which was proved in early 1970's.

We have seen that symmetries give rise to conservation laws: that is, functions that are constant along the trajectories of our system. We'll put these scalar valued functions together into one vector valued function called the moment(um) map. In the case where the group of symmetries are rotations, this is simply angular momentum. For translational symmetries it's linear momentum.

The key idea is to fix a (regular) value of the moment map, which singles out a submanifold of our phase space preserved under dynamics and then ~~to~~ divide out the symmetries.

The theorem of Marsden-Weinstein-Meyer asserts that the result is a new Hamiltonian system, which has fewer degrees of freedom. In the case of translational symmetrics the theorem amounts to passing to the center of mass coordinates. This and other special cases were known to 19th century mathematicians and physicists. Jacobi's "elimination of nodes" is an example.

We get to work by reviewing our general set-up.

We have a physical system with the configuration space M, which we take to be a manifold. There is an action of a Lie group G on M, the symmetries of the system. This action lifts to an action of G on TM and on T^*M. The dynamics is given by a Lagrangian $L: TM \to \mathbb{R}$, which we assume to be <u>G-invariant</u>. That is, $\forall a \in G$, $\forall q \in M$, $\forall v \in T_q M$,

$$L(a\cdot(q,v)) = L(q,v).$$

We further <u>assume</u>: the Legendre transform $\mathcal{L}_L: TM \to T^*M$ is a diffeomorphism. This is true, for example, if L is of the form

$$L(q,v) = \frac{1}{2} g_q(v,v) + V(x)$$

where g is some Riemannian metric on M and $V: M \to \mathbb{R}$ is a smooth function, a potential.

<u>Exercise</u>: 1) Prove that if $L: TM \to \mathbb{R}$ is an invariant Lagrangian, then the Legendre transform $\mathcal{L}_L: TM \to T^*M$ is equivariant:

$$\mathcal{L}_L(a\cdot(q,v)) = a \cdot \mathcal{L}_L(q,v)$$

for all $a \in G$, all $(q,v) \in TM$

2) Prove that the corresponding Hamiltonian

$$H(q,p) = \langle p, \mathcal{L}_L^{-1}(q,p)\rangle - L(\mathcal{L}_L^{-1}(q,p))$$

is G-invariant.

<u>Moment maps</u>

<u>Recall</u>: 1) if a Lie group G acts on a manifold M, then for any $X \in \mathfrak{g} = \mathrm{Lie}(G)$ the induced vector field X_{T^*M} on T^*M is Hamiltonian:

$$\iota(X_{T^*M})(d\alpha) = -d\mu^X \qquad \text{where}$$

$$\mu^X(q,p) = \langle p, X_M(q)\rangle$$

2) For any G-invariant Hamiltonian $H: TM \to \mathbb{R}$ the functions μ^X are "invariants of motion" ("constants of motion"):

for any trajectory $(q(t), p(t))$ solving

$$\frac{d}{dt}(q(t), p(t)) = X_H(q(t), p(t))$$

we have

$$\mu^X(q(t), p(t)) = \mu^X(q(0), p(0)) \qquad \forall t.$$

We put the functions $\{\mu^X\}_{X \in \mathfrak{g}}$ together into one vector-valued function $\mu: T^*M \to \mathfrak{g}$ as follows:

for any $(q,p) \in T^*M$, $\mu(q,p): \mathfrak{g} \to \mathbb{R}$ is the unique linear map so that

$$(*) \qquad \langle \mu(q,p), X\rangle = \mu^X(q,p)$$

for all $X \in \mathfrak{g}$.

The function $\mu: T^*M \to \mathfrak{g}$ defined by $(*)$ is called a <u>moment map</u>. The term <u>momentum map</u> is also used.

<u>Example</u> Consider the action of $G = \mathbb{R}^3$ on $\mathbb{R}^3 \times \mathbb{R}^3$ by translations:

$$v \cdot (\vec{r}_1, \vec{r}_2) = (\vec{r}_1 + v, \vec{r}_2 + v)$$

for all $v \in G$, $\vec{r}_1, \vec{r}_2 \in \mathbb{R}^3$.

$Lie(\mathbb{R}^3) = \mathbb{R}^3$ and $\exp: Lie(\mathbb{R}^3) \to \mathbb{R}^3$ is the identity. Hence, for any $X \in \mathbb{R}^3$

$$X_{(\mathbb{R}^3)^2}(\vec{r}_1, \vec{r}_2) = \frac{d}{dt}\Big|_0 (\exp tX)\cdot(\vec{r}_1, \vec{r}_2) = \frac{d}{dt}\Big|_0 (\vec{r}_1 + tX, \vec{r}_2 + tX) = (X, X).$$

Hence the corresponding moment map

$$\mu: T^*(\mathbb{R}^3)^2 \to (\mathbb{R}^3)^*$$

is defined by

$$\langle \mu((\vec{r}_1, \vec{r}_2), (\vec{p}_1, \vec{p}_2)), X\rangle = \langle (\vec{p}_1, \vec{p}_2), (X, X)\rangle = \langle \vec{p}_1, X\rangle + \langle \vec{p}_2, X\rangle = \langle p_1 + p_2, X\rangle$$

$$\therefore \quad \mu((\vec{r}_1, \vec{r}_2), (\vec{p}_1, \vec{p}_2)) = \vec{p}_1 + \vec{p}_2.$$

$\square$

<u>Exercise</u> Consider n particles in $\mathbb{R}^3$ with masses $m_1 \dots m_n$, interacting by way of a translation-invariant potential $V: (\mathbb{R}^3)^n \to \mathbb{R}$. Write down the Lagrangian for the system, compute the corresponding Hamiltonian. Next show that

$$\mu: T^*((\mathbb{R}^3)^n) \simeq (\mathbb{R}^3)^n \times ((\mathbb{R}^3)^*)^n \to (\mathbb{R}^3)^*$$

given by

$$\mu(\vec{r}_1, \dots \vec{r}_n, \vec{p}_1, \dots \vec{p}_n) = \sum_{i=1}^n \vec{p}_i$$

is the moment map for the action of $\mathbb{R}^3$. Explain why it gives the constants of motion.

As before, suppose a Lie group G acts on a manifold M. Next suppose $H: T^*M \to \mathbb{R}$ is G-invariant. We argue that

(1) the Hamiltonian vector field X_H of H is G-invariant

and that consequently

(2) if $(q(t), p(t))$ is a trajectory of X_H, then so is $a \cdot (q(t), p(t))$ for any $a \in G$.

We start with a slightly more general set-up: a group G acting on a manifold Q. Recall then that $\forall a \in G$ we have a diffeomorphism

$$a_Q: Q \to Q, \quad a_Q(q) := a \cdot q.$$

<u>Definition</u> A vector field $X: Q \to TQ$ is <u>G-invariant</u> if

$$(da_Q)_q \, X(q) = X(a \cdot q)$$

for all $q \in Q$, $\forall a \in G$.

<u>Proposition</u> Suppose $X: Q \to TQ$ is G-invariant and $\gamma(t)$ is a trajectory of X; that is $\frac{d}{dt}\gamma(t) = X(\gamma(t))$. Then $\forall a \in G$ $a \cdot \gamma(t)$ is also a trajectory of X.

Proof We compute:

$$\frac{d}{dt}\, a\cdot\gamma(t) = \frac{d}{dt}\, a_Q(\gamma(t)) \overset{\text{chain rule}}{=} (da_Q)_{\gamma(t)}\left(\frac{d\gamma}{dt}\right) = (da_Q)_{\gamma(t)} X(\gamma(t)) =$$
$$= X(a\cdot\gamma(t)) \quad \text{since } X \text{ is } G\text{-invariant}$$

$\therefore$ $a\cdot\gamma(t)$ is also a trajectory of X $\square$.

We next argue that if $H: T^*M \to \mathbb{R}$ is a G-invariant Hamiltonian then its Hamiltonian vector field X_H is also G-invariant. Consequently if $(q(t), p(t))$ solves $\frac{d}{dt}(q,p) = X_H(q,p)$ then so does $a\cdot(q(t), p(t))$ for all $a \in G$.

Recall 1. The lifted action of G on T^*M preserves the tautological 1-form $\alpha \in \Omega^1(T^*M)$: $\forall a \in G$

$$(a_{T^*M})^*\alpha = \alpha$$

2. For any C^∞ map $f: N \to N'$ and any k-form $\tau \in \Omega^k(N')$.

$$f^*(d\tau) = d(f^*\tau).$$

(1) + (2) $\Rightarrow$ (i) for $\omega = -d\alpha$,

$$(a_{T^*M})^*\omega = (a_{T^*M})^*(-d\alpha) = -d(a_{T^*M})^*\alpha = -d\alpha = \omega.$$

(ii) Since H is G-invariant

$$H\circ a_{T^*M} = H \quad \text{, ie} \quad H = (a_{T^*M})^* H.$$

$$\to \quad dH = d((a_{T^*M})^* H) = (a_{T^*M})^* dH.$$

$\Rightarrow$ dH is G-invariant.

Proposition 2 If $H \in C^\infty(T^*M)$ is G-invariant, so is its Hamiltonian vector field X_H.

Proof Recall that X_H is defined by

$$\omega_{(q,p)}(X_H(q,p), v) = -(dH)_{(q,p)}(v)$$

for all $(q,p) \in T^*M$, all $v \in T_{(q,p)}(T^*M)$.

We want to show: $\quad da_{T^*M}(X_H(q,p)) = X_H(a_{T^*M}(q,p)) \quad \forall a \in G$
$\forall (q,p) \in T^*M$.

Since ω is non-degenerate, it's enough to show that

$$(+) \quad \omega_{a\cdot(q,p)}\left(da_{T^*M}(X_H(q,p)), w \right) = \omega_{a\cdot(q,p)}\left(X_H(a\cdot(q,p)), w \right)$$

for all $w \in T_{a\cdot(q,p)}(T^*M)$.

Since $(da_{T^*M}) : T_{(q,p)}(T^*M) \to T_{a\cdot(q,p)}(T^*M)$ is an isomorphism, we may assume $w = (da_{T^*M})(v)$ for some $v \in T_{(q,p)}(T^*M)$.

So we want to prove: $\forall a \in G$, $\forall (q,p) \in T^*M$, $\forall v \in T_{(q,p)}(T^*M)$

$$(\#) \quad \omega_{a\cdot(q,p)}\left(da_{T^*M}(X_H(q,p)), da_{T^*M}\, v \right) = \omega_{a\cdot(q,p)}\left(X_H(a\cdot(q,p)), da_{T^*M}(v) \right)$$

Left hand side of (#) is $((a_{T^*M})^*\omega)_{(q,p)}(X_H(q,p), v) = \omega_{(q,p)}(X_H(q,p), v)$
$= -(dH)_{(q,p)}(v)$

Right hand side of (#) is

$$-(dH)_{a\cdot(q,p)}\left(da_{T^*M}(v) \right) = -\left((a_{T^*M})^* dH\right)_{(q,p)}(v) \underset{\uparrow}{=} -(dH)_{(q,p)}(v)$$

since dH is G-invariant.

$\Rightarrow$ (#) holds.

$\Rightarrow$ (+) holds

$\Rightarrow$ X_H is G-invariant. $\square$

$\Rightarrow$ If $(q(t), p(t))$ is a trajectory of X_H then so is $a\cdot(q(t), p(t))$ for any $a \in G$.

Orbit spaces

Consider an action of a Lie group G on a manifold N.

Definition An orbit of G through $x \in N$ is the set

$$G\cdot x := \{ g\cdot x \mid g \in G \}.$$

Exercise (important!) If $G\cdot x \cap G\cdot y \neq \emptyset$ for some $x, y \in N$ then $G\cdot x = G\cdot y$

Consequence The orbits partition N into disjoint sets.

Definition The orbit space N/G for an action of a Lie group G on a manifold N is the set

$$N/G := \{ G\cdot x \mid x \in N \},$$

the set of orbits.

Remark N/G can be made into a topological space but in general it may not be a manifold.

Example $SO(2)$ acts on $\mathbb{R}^2$ by rotations. The orbits are $\{0\}$ and circles

$$C_r = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = r^2\}$$

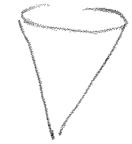
By exercise every point in $\mathbb{R}^2$ lies on a unique orbit: $(0,0) \neq (x,y) \in \mathbb{R}^2$ lies on the circle of radius $r = \sqrt{x^2+y^2}$.

The space of orbits is the ray $[0, \infty)$.

And we have a well defined orbit map $\pi : \mathbb{R}^2 \to [0,\infty)$,

$$\pi(x,y) = \sqrt{x^2+y^2}$$

Example $G = \{\pm 1\}$ acts on $\mathbb{R}^2$ by $\varepsilon \cdot (x,y) = (\varepsilon x, \varepsilon y)$, $\varepsilon = \pm 1$.

The space of orbits is a cone:  ;

every orbit is uniquely represented by (x,y) with $x \geq 0$ except for the points on the y-axis: $(0,y)$ and $(0,-y)$ lie on the same orbit.

In good cases the space of orbits N/G is a manifold.

Example $\mathbb{R}^3$ acts on $(\mathbb{R}^3)^n$ by translations.

Claim $(\mathbb{R}^3)^n / \mathbb{R}^3 \simeq (\mathbb{R}^3)^{n-1}$

Reason: every orbit $\mathbb{R}^3 \cdot \begin{pmatrix} \vec{r}_1 \\ \vdots \\ \vec{r}_n \end{pmatrix}$ is uniquely determined by a vector in $(\mathbb{R}^3)^n$ with the first "coordinate" $\vec{0}$:

$$\mathbb{R}^3 \cdot \begin{pmatrix} \vec{r}_1 \\ \vdots \\ \vec{r}_n \end{pmatrix} = \mathbb{R}^3 \cdot \begin{pmatrix} \vec{0} \\ \vec{r}_2 - \vec{r}_1 \\ \vdots \\ \vec{r}_n - \vec{r}_1 \end{pmatrix}$$

Example Let G be a Lie group. It acts on itself by left multiplication: $a \cdot x = ax =: L_a(x)$ $\forall a, x \in G$.

We get a lifted action of G on T^*G.

Claim $T^*G/G \simeq \mathfrak{g}^*$

Reason/proof $\forall q \in G$, $L_q(e) = qe = e \Rightarrow (dL_q)_e : \mathfrak{g} = T_eG \xrightarrow{\simeq} T_qG$

$(dL_q)^T : T_q^*G \to T_e^*G = \mathfrak{g}^*$.

$\Rightarrow \forall q \in G$, $\forall p \in T_q^*G$, $(dL_q)_q^T p \in T_e^*G = \mathfrak{g}^*$

It follows that

$$G \cdot (q, p) = G \cdot (e, (dL_q)_q^T p)$$

Consequently the map

$$T^*G/G \to \mathfrak{g}^*, \quad G \cdot (q, p) \longmapsto (dL_q)_q^T p$$

is a bijection and T^*G/G "is" $\mathfrak{g}^*$.

Adjoint and coadjoint actions.

A Lie group G acts on itself by conjugation: for any $a \in G$ we have a map $C_a : G \to G$, $C_a(x) = axa^{-1}$.

Since $C_a(e) = aea^{-1} = e$, its differential $(dC_a)_e$ maps T_eG to itself. This gives us an action of G on $\mathfrak{g} = T_eG$:

$\forall a \in G$ $\forall X \in \mathfrak{g}$ $\quad Ad(a)X := (dC_a)_e X$

[Aside check that this is an action: $\forall a, b \in G$, $X \in \mathfrak{g}$

$$Ad(ab)X = Ad(a)(Ad(b)X)$$

Since each map $Ad(a) : \mathfrak{g} \to \mathfrak{g}$ is linear (and invertible), putting them together gives us a map

$$Ad : G \to GL(\mathfrak{g}), \quad a \mapsto Ad(a).$$

It's called the adjoint representation.

Dually we have the coadjoint representation

$$Ad^+ : G \to GL(\mathfrak{g}^*).$$

It's defined by

$$\mathrm{Ad}^+(a)\,\ell = \ell \circ \mathrm{Ad}(a^{-1})$$

for all $a \in G$, $\ell \in \mathfrak{g}^*$.

<u>Proposition</u> The moment map
$$\mu : T^*M \to \mathfrak{g}^*$$
arising from an action of a Lie group G on a manifold M is <u>equivariant</u>. That is
$$\mu(a \cdot (q,p)) = \mathrm{Ad}^+(a)\,\mu(q,p)$$
for all $(q,p) \in T^*M$, $\forall a \in G$.

<u>Proof</u> This is not hard, but it requires a fact we haven't proved: $\forall a \in G$, $\forall X \in \mathfrak{g}$ $\mathbb{R}$
$$a \exp(X)\, a^{-1} = \exp(\mathrm{Ad}(a)X).$$
< ask me if you want to see the full proof >

We need two more definitions before we can state Marsden-Weinstein-Meyer reduction theorem.

<u>Definition</u> Suppose a group G acts on a manifold N. The <u>stabilizer</u> of $x \in N$ is the set
$$G_x := \{ g \in G \mid g \cdot x = x \}$$
of elements of G that <u>fix</u> x.

<u>Aside</u> G_x is a group.

<u>Definition</u> An action of a group G on a manifold N is <u>free</u> if $G_x = \{e\}$ for all $x \in N$, i.e.
$$g \cdot x = x \Rightarrow g = e \quad \text{for all } x \in N.$$

A slightly simplified version of Marsden-Weinstein-Meyer theorem reads:

Theorem Suppose a Lie group G acts on a manifold M.
Let $\mu: T^*M \to \mathfrak{g}^*$ be the corresponding moment map.
Fix $\alpha \in \mathfrak{g}^*$ and suppose G_α acts freely on $\mu^{-1}(\alpha)$.
[G_α acts on $\mu^{-1}(\alpha)$ since $\forall\, x \in \mu^{-1}(\alpha)$, $\forall\, a \in G_\alpha$
$\mu(a \cdot x) = a \cdot \mu(x) = a \cdot \alpha = \alpha$]

Then

(1) α is a regular value of μ and $\mu^{-1}(\alpha)$ is a manifold.

Moreover

(2) $M_\alpha := \mu^{-1}(\alpha) / G_\alpha$ is naturally a symplectic manifold.

Finally, for any G-invariant Hamiltonian $H \in C^\infty(T^*M)$ the flow of X_H not only preserves $\mu^{-1}(\alpha)$ but also descends to the flow of a Hamiltonian vector field for a naturally defined Hamiltonian $H_\alpha : M_\alpha \to \mathbb{R}$.

(Congratulations. You survived the course.)