1. The tangent bundle

Definition 1.1 (provisional)**.** The *tangent bundle* TM of a manifold M is (as a set)

$$TM = \bigsqcup_{a\in M} T_aM.$$

Note that there is a natural projection (the tangent bundle projection)

$$\pi : TM \to M$$

which sends a tangent vector $v \in T_aM$ to the corresponding point a of M.

We want to show that the tangent bundle TM itself is a manifold in a natural way and the projection map $\pi : TM \to M$ is smooth. Strictly speaking, we first should specify a topology on TM. However, our strategy will be different. We will first find candidates for coordinate charts on the tangent bundle TM. They will be constructed out of coordinate charts on M. We will check that the change of these candidate coordinates on TM is smooth. One then can use these candidate coordinates to manufacture a topology on TM, an issue that we will ignore.

Remark 1.2 (Comment on notation)**.** On $\mathbb{R}^n$ we have n real valued smooth coordinate functions $r_1, \dots, r_n$; r_j assigns to a point $a = (a_1, \dots, a_n)$ its jth coordinate: $r_j(a_1, \dots, a_n) = a_j$. If $F : X \to \mathbb{R}^n$ is any function then F cooresponds to an n-tuple of real-valued functions on X:

$$F(x) = (F_1(x), \dots, F_n(x),$$

where $F_j(x)$ is the jth coordinate of $F(x)$. Thus $F_j(x) = r_j(F(x)) = (r_j\circ F)(x)$ and consequently $F_j = r_j\circ F$.
Similarly, but not entirely consistatntly, we say "$\varphi = (x_1, \dots x_n) : U \to \mathbb{R}^n$ is a coordinate chart on a manifold M" to mean that $x_j(m) = r_j(\varphi(m))$ for $m \in U \subset M$. Thus $x_j = r_j \circ \varphi$.

Let $\phi = (x_1, \cdots , x_n) : U \to \mathbb{R}^n$ be a coordinate chart on M. Out of it we construct a chart on $TU = \bigsqcup_{a\in U} T_aM \subset TM$. The first n functions come for free: we take the functions $x_1 \circ \pi, \dots , x_n \circ \pi$. Another set of n functions come for free also: given a vector $v \in T_aU$,

$$v = \sum (dx_i)_a(v)\frac{\partial}{\partial x_i}|_a.$$

Hence, abusing the notation a bit, we get maps

$$dx_i : TU \to \mathbb{R}, \quad TU \ni v \mapsto (dx_i)_a(v), \text{ where } a = \pi(v).$$

Thus we define a candidate coordinate chart

$$\tilde{\phi} := (x_1 \circ \pi, \cdots , x_n \circ \pi, dx_1, \cdots , dx_n) : TU \to \mathbb{R}^n \times \mathbb{R}^n$$

by

$$\tilde{\phi}(v) = (x_1(\pi(v)), \dots , x_n(\pi(v)), (dx_1)_{\pi(v)}(v), \dots , (dx_n)_{\pi(v)}(v)).$$

If $\{U_\alpha, \phi_\alpha)\}$ is an atlas on M, we get a candidate atlas $\{(TU_\alpha, \tilde{\phi}_\alpha)\}$ on TM. To see why this could possibly be an atlas, we need to check that the change of coordinates in this new purported atlas is smooth. To this end pick two coordinate charts $(U, \phi = (x_1, \cdots , x_n))$ and $(V, \psi = (y_1, \cdots , y_n))$ on M with $U \cap V \neq \emptyset$. Then $T(U \cap V) = TU \cap TV \neq \emptyset$. Let

$$\tilde{\phi} = (x_1, \cdots , x_n, dx_1, \cdots , dx_n) : TU \to \mathbb{R}^n \times \mathbb{R}^n$$

and

$$\tilde{\psi} = (y_1, \cdots , y_n, dy_1, \cdots , dy_n) : TV \to \mathbb{R}^n \times \mathbb{R}^n$$

be the corresponding candidates charts on TM. Now let us compute the change of coordinates $\tilde{\psi} \circ \tilde{\phi}^{-1}$.
First, note that

$$\tilde{\phi}^{-1}(r_1, \cdots , r_n, u_1, \cdots , u_n) = \sum_i u_i \frac{\partial}{\partial x_i}\Big|_{\phi^{-1}(r_1,\cdots,r_n)} \in T_{\phi^{-1}(r_1,\cdots,r_n)}M.$$

So

$$\tilde{\psi}(\sum u_i \frac{\partial}{\partial x_i}\Big|_{\phi^{-1}(r_1,\cdots,r_n)}) = (\psi(\phi^{-1}(r_1, \cdots , r_n)), dy_1(\sum_i u_i \frac{\partial}{\partial x_i}), \cdots , dy_n(\sum_i u_i \frac{\partial}{\partial x_i})).$$

But

$$dy_j(\sum_i u_i \frac{\partial}{\partial x_i}) = \sum_i u_i(\frac{\partial}{\partial x_i}(y_j)) = \sum_i \frac{\partial y_j}{\partial x_i} u_i = \sum_i \frac{\partial}{\partial r_i}(r_j(\psi \circ \phi^{-1}))u_i.$$

Thus the change of the candidate coordinates is given by

$$\begin{aligned}\tilde{\psi} \circ \tilde{\phi}^{-1}(r_1, \cdots, r_n, u_1, \cdots, u_n) =& (\psi \circ \phi^{-1}(r), (\sum_i \frac{\partial y_1}{\partial x_i}(r)u_i, \ldots, \sum_i \frac{\partial y_n}{\partial x_i}(r)u_i)) \\ =& (\psi \circ \phi^{-1}(r), \left(\frac{\partial y_j}{\partial x_i}(r)\right) \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}),\end{aligned} \tag{1.1}$$

where $r = (r_1, \ldots r_n)$. Clearly $\tilde{\psi} \circ \tilde{\phi}^{-1}$ is smooth wherever it is defined. It remains to define a topology on TM so that the charts $\tilde{\phi} : TU \to \phi(U) \times \mathbb{R}^n$ are homeomorphisms. We declare a subset $O \subset TM$ to be open if for any coordinate chart $\phi : U \to \mathbb{R}^n$ on M, the set $\tilde{\phi}(O \cap TU) \subset \mathbb{R}^n \times \mathbb{R}^n$ is open.

Proposition 1.3. *The collection of open sets on TM defined above does indeed form a topology. Moreover, if M is Hausdorff and second countable, so is TM.*

Proof. We ignore these technicalities and save them for a graduate course in differential topology. □

We conclude that if M is an n-dimensional then its tangent bundle TM is a $2n$-dimensional manifold. Moreover, each coordinate chart $(x_1, \ldots x_n) : U \to \mathbb{R}^n$ on M gives rise to a coordinate chart $(x_1 \circ \pi, \ldots x_n \circ \pi, dx_1, \ldots, dx_n) : TU \to \mathbb{R}^{2n}$.

Remark 1.4. The following notation is suggestive: we write $(m, v) \in TM$ for $v \in T_m(M)$. Strictly speaking, it is redundant since $m = \pi(v)$.

Remark 1.5. It is customary to simply write $x_i : TU \to \mathbb{R}$ for $x_i \circ \pi : TU \to \mathbb{R}$.

2. The cotangent bundle

As a set, the cotangent bundle T^*M is the disjoint union of cotangent spaces:

$$T^*M = \bigsqcup_{a \in M} T_a^*M.$$

Note that there is a natural projection (the cotangent bundle projection)

$$\pi : T^*M \to M$$

which sends a cotangent vector (a covector for short) $\eta \in T_a^*M$ to the corresponding point a of M. We make the cotangent bundle T^*M into a manifold in more or less the same way we made the tangent bundle into a manifold. That is, we manufacture new coordinate charts on T^*M out of coordinate charts on M and check that the transition maps between the new coordinate charts are smooth.

So let $\phi = (x_1, \ldots, x_n) : U \to \mathbb{R}^n$ be a coordinate chart on M. Then for each point $a \in U$ the covectors $\{(dx_i)_a\}$ form a basis of T_a^*M. The partials $\{\frac{\partial}{\partial x_i}|_a\}$ form the dual basis. Hence for any $\eta \in T_a^*M$,

$$\eta = \sum \eta(\frac{\partial}{\partial x_i}|_a)\,(dx_i)_a.$$

Therefore the partials $\{\frac{\partial}{\partial x_i}\}$ give us coordinate functions on T^*U:

$$\frac{\partial}{\partial x_i} : T^*U \to \mathbb{R}^n, \quad T^*U \ni \eta \mapsto \eta(\frac{\partial}{\partial x_i}|_a),$$

where $a = \pi(\eta)$. We now define the candidate coordinates

$$\bar{\phi} : T^*U \to \mathbb{R}^n \times \mathbb{R}^n$$

by

$$\bar{\phi} = (x_1 \circ \pi, \ldots, x_n \circ \pi, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}).$$

Note that

$$\bar{\phi}^{-1}(r_1, \dots, r_n, w_1, \dots, w_n) = \sum_{i=1}^{n} w_i (dx_i)_{\phi^{-1}(r)} \in T^*_{\phi^{-1}(r)} M,$$

where again we have abbreviated $(r_1, \dots, r_n)$ as r. We now check the transition maps. Let $\psi = (y_1, \dots, y_n) : V \to \mathbb{R}^n$ be a coordinate chart on M with $V \cap U \neq \emptyset$. Then

$$\begin{aligned}\bar{\psi} \circ \bar{\phi}^{-1}(r_1, \dots, r_n, w_1, \dots, w_n) =& \bar{\psi}(\sum_{i=1}^{n} w_i (dx_i)_{\phi^{-1}(r)}) \\ =& ((\psi \circ \phi^{-1})(r), \frac{\partial}{\partial y_1}(\sum_{i=1}^{n} w_i \, dx_i), \dots, \frac{\partial}{\partial y_n}(\sum_{i=1}^{n} w_i \, dx_i)) \\ =& ((\psi \circ \phi^{-1})(r), \sum_i w_i \frac{\partial x_i}{\partial y_1}, \dots, \sum_i w_i \frac{\partial x_i}{\partial y_n}).\end{aligned}$$

We conclude that

$$\bar{\psi} \circ \bar{\phi}^{-1}(r_1, \cdots, r_n, w_1, \cdots, w_n) = (\psi \circ \phi^{-1}(r), \left(\frac{\partial x_i}{\partial y_j}(r)\right) \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}), \tag{2.1}$$

which is smooth. The rest of the argument proceeds as in the case of the tangent bundle.

Remark 2.1. Later on, when we look at the general vector bundles, it will be instructive to compare the formulas for the change of coordinates in the tangent and the cotangent bundles. In particular note that the matrices $\left(\frac{\partial y_j}{\partial x_i}(r)\right)$ and $\left(\frac{\partial x_i}{\partial y_j}(r)\right)$ are inverse transposes of each other.