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10.1

Recall A topological space is a set $X$ with a topology $\mathcal{T}$.

$\mathcal{T}$ is a collection of subsets of $X$ (called open sets) s.t.

1) $\emptyset, X \in \mathcal{T}$

2) if $U, V \in \mathcal{T}$ Then $U \cap V \in \mathcal{T}$

3) if $\{U_\alpha\}_{\alpha \in A}$ is a collection of open sets Then $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}$

$\mathbb{R}^n$ comes with a standard topology $\mathcal{T}$:

$U \in \mathcal{T}$ if $\forall x \in U$ $\exists$ open ball $B_r^d(x)$ s.t. $B_r^d(x) \subseteq U$.

Here $d = d_2, d_1$ or $d_\infty$; all of these metrics define the same topology on $\mathbb{R}^n$.

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Given a topological space $(X, \mathcal{T})$ and a subset $K \subseteq X$ an open cover of $K$ is a collection $\{\mathcal{O}_\alpha\}_{\alpha \in A}$ of open sets s.t.

$$\bigcup_{\alpha \in A} \mathcal{O}_\alpha \supseteq K.$$

Definition A subset $K$ of a topological space $X = (X, \mathcal{T})$ is compact if for every open cover $\{U_\alpha\}_{\alpha \in A}$ of $K$ there is a finite subcover, i.e. $\exists \alpha_1, \dots \alpha_n \in A$ s.t.

$$K \subseteq U_{\alpha_1} \cup \dots \cup U_{\alpha_n}.$$

Ex Any finite set $K$ is compact: If $\{U_\alpha\}_{\alpha \in A}$ is an open cover of $K = \{x_1, \dots x_n\}$ Then $\forall i$, $x_i \in U_{\alpha_i}$ for some $\alpha_i \in A$ and consequently $K = \bigcup_{i=1}^n \{x_i\} \subseteq \bigcup_{i=1}^n U_{\alpha_i}$.

(if $\alpha_i = \alpha_j$ for some $i \neq j$, That's fine too).

Non Ex $\mathbb{R}$ is not compact: $\{(n, n+2)\}_{n \in \mathbb{Z}}$ is an open cover of $\mathbb{R}$, but we can't drop any of $(n, n+2)$ and still have a cover.

$\mathbb{N} \subset \mathbb{R}$ is not compact: $\{(n-1/2, n+1/2)\}_{n\in\mathbb{N}}$ is an open cover with no finite subcover.

Ex $E$ = a set $d(x,y) = 1$ for $x \neq y$, $T_d$ corr. topology.

$K \subseteq E$ is compact $\Leftrightarrow$ $K$ is finite

Reason $\forall x \in E$, $\{x\} = B_{1/2}(x)$ is open.

$\Rightarrow \{\{x\}\}_{x\in K}$ is an open cover of $K$

It has no proper subcovers and it's finite $\Leftrightarrow$ $K$ is finite.

Lemma 10.1 $(X,T)$ top space, $K \subseteq X$ compact, $C \subseteq X$ closed.

Then $K \cap C$ is compact.

Proof Suppose $\{U_\alpha\}_{\alpha\in A}$ is an open cover of $K\cap C$: $K \cap C \subseteq \bigcup_{\alpha\in A} U_\alpha$

Then $\{U_\alpha\}_{\alpha\in A} \cup \{X \setminus C\} \supseteq (K\cap C) \cup (X \setminus C) \supseteq K$

Since $K$ is compact, $\exists n$, $\alpha_1 \dots \alpha_n$ st

$$(K \cap C) \cup (K \setminus C) \subseteq U_{\alpha_1} \cup \dots \cup U_{\alpha_n} \cup (X \setminus C)$$

$$\Rightarrow K \cap C \subseteq U_{\alpha_1} \cup \dots \cup U_{\alpha_n}.$$

$\therefore$ $K \cap C$ is compact.

Thm 10.2 Let $(E,d)$ be a metric space. If $K \subseteq E$ is compact (w.r.t. $T_d$) then $K$ is closed and bounded.

Proof ($K$ is bounded) Choose any $x \in E$.

Then $E = \bigcup_{n=1}^{\infty} B_n(x)$.

$\Rightarrow$ $\{B_n(x)\}_{n\in\mathbb{N}}$ is an open cover of $K$.

Since $K$ is compact, there is a finite subcover

$B_{n_1}(x), \dots B_{n_k}(x)$ of $K$

May assume $n_1 < n_2 < \dots < n_k$. And then

$$K \subseteq B_{n_1}(x) \cup \dots \cup B_{n_k}(x) \subseteq B_{n_k}(x)$$

$\therefore$ $K$ is bounded.

(K is closed) If $K = E$, we're done, since $E \setminus K = \emptyset$ is open.
Suppose $K \neq E$, and $x \in E \setminus K$.
$\forall r > 0$ $\quad \bar{B}_r(x) := \{y \in E \mid d(x,y) \le r\}$ is closed.
$\Rightarrow U_r := E \setminus \bar{B}_r(x)$ is open.

$$\bigcup_{r>0} U_r = \bigcup_{r>0} \left(E \setminus \bar{B}_r(x)\right) = E \setminus \bigcap_{r>0} \bar{B}_r(x) = E \setminus \{x\}.$$

Since $K \subseteq E \setminus \{x\}$, $\{U_r\}_{r>0}$ is an open cover of $K$.
Since $K$ is compact $\exists\, r_1 < r_2 < \ldots < r_k$ s.t. $K \subseteq U_{r_1} \cup \ldots \cup U_{r_k}$
$= E \setminus \left(\bar{B}_{r_1}(x) \cap \ldots \cap \bar{B}_{r_k}(x)\right) = E \setminus \bar{B}_{r_1}(x)$.
Then $B_{r_1}(x) \cap K = \emptyset$. $\Rightarrow$ $E \setminus K$ is open. ∎

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WARNING $d(x,y) = \max\{1, |x-y|\}$ is a metric on $\mathbb{R}$.
In this metric $\mathbb{R}$ is bounded (and closed)
$T_d$ = standard topology on $\mathbb{R}$.
So $\mathbb{R}$ is not compact.
Moral closed + bounded $\not\Rightarrow$ compact (in general)

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We'll see: in $(\mathbb{R}^n, d_2)$ (or $d_1$ or $d_\infty$)
compact $\Leftrightarrow$ closed and bounded.

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Remark In general compact sets don't need to be closed.
Ex $X = \{a, b\}$, $T = (X, \emptyset, \{a\})$
$K = \{a\}$ is compact (since it's finite); it's not closed
since $X \setminus K = \{b\} \notin T$.

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Theorem 10.3 Let $X$ be a topological space,
$K_1 \supseteq K_2 \supseteq \ldots \supseteq K_n \supseteq \ldots$
a sequence of closed compact sets (they are automatically

closed if the topology comes from a metric)

Then $K := \bigcap_{n=1}^{\infty} K_n \neq \emptyset$ and compact.

Proof Since $K$ is closed (why?) and since $K \subseteq K_1$, it's compact.

We argue $K \neq \emptyset$. Suppose not: $\bigcap_{n=1}^{\infty} K_n = \emptyset$.

Then $E = E \setminus \bigcap_{n=1}^{\infty} K_n = \bigcup_{n=1}^{\infty} (E \setminus K_n)$

$\Rightarrow \{U_n := E \setminus K_n\}_{n=1}^{\infty}$ is an open cover of $E$ hence of $K_1$

Since $K_1$ is compact, $\exists\, n_1 < n_2 < \dots < n_\ell$ s.t.

$$K_1 \subseteq (E \setminus K_{n_1}) \cup (E \setminus K_{n_2}) \dots \cup (E \setminus K_{n_\ell})$$
$$= E \setminus \bigcap_{i=1}^{\ell} K_{n_i} =$$

But $K_{n_1} \supseteq K_{n_2} \supseteq \dots \supseteq K_{n_\ell}$. So $K_1 \subseteq E \setminus K_{n_\ell}$

Contradiction since $K_{n_\ell} \subseteq K_1$. ◻

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Note • $U_n = (0, 1/n)$, $n = 1, 2, \dots$ is a nested sequence of open sets with $\bigcap_{n=1}^{\infty} U_n = \emptyset$. $U_n$'s cannot be compact since they are not closed in $\mathbb{R}$.

• $V_n = [n, \infty)$ is a nested sequence of closed sets in $\mathbb{R}$ $\bigcap_{n=1}^{\infty} V_n = \emptyset$. Again, $V_n$'s are not compact since they're not bounded.

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Definition A subset $K$ of a topological space is sequentially compact if every sequence in $K$ has a convergent subsequence with a limit in $K$.

Ex $K \subseteq \mathbb{R}^n$ closed and bounded, $\{s_n\} \subset K$ a sequence. By HW#3, problem 6 it has a convergent subsequence $\{s_{n_k}\}$. Since $K$ is closed, $\lim s_{n_k} \in K$. $\Rightarrow K$ is sequentially compact.