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11.1

Last time: In a metric space (E,d), $K \subseteq E$ compact $\Rightarrow$ K is closed and bounded. (The converse is false)

Theorem 10.3 X topological space, $K_1 \supseteq K_2 \supseteq \dots \supseteq K_n \supseteq \dots$ sequence of nested closed compact sets. Then $\bigcap_{n=1}^{\infty} K_n \neq \emptyset$.

Proof Suppose $\bigcap_{n=1}^{\infty} K_n = \emptyset$.

Then

$$E = E \setminus \bigcap_{n=1}^{\infty} K_n = \bigcup_{n=1}^{\infty} (E \setminus K_n) \Rightarrow \{E \setminus K_n\}_{n=1}^{\infty} \text{ is an open}$$

cover of E and, hence, of K_1. Since K_1 is compact

$\exists$ $n_1 \leq n_2 \leq \dots \leq n_e$ st

$$K_1 \subseteq (E \setminus K_{n_1}) \cup (E \setminus K_{n_2}) \cup \dots \cup (E \setminus K_{n_e}) = E$$

$$= E \setminus (K_{n_1} \cap K_{n_2} \cap \dots \cap K_{n_e}) \underset{\uparrow}{=} E \setminus K_{n_e}$$

K_i's are nested!

Contradiction since $K_{n_e} \subseteq K_1$. ▷

Definition A subset K of a topological space is sequentially compact if every sequence in K has a convergent subsequence with a limit in K.

$\boxed{\text{Ex 11.0}}$ Suppose $K \subseteq \mathbb{R}^n$ is closed and bounded. Then K is sequentially compact: by HW#3 problem 6 any sequence $\{s_n\}$ in K has a convergent subsequence $\{s_{n_k}\}$. Since K is closed, $L = \lim_{k \to \infty} s_{n_k} \in K$.

Lemma 11.1 Let (E,d) be a metric space. Suppose $K \subseteq E$ is compact. Then K is sequentially compact.

Proof Let $\{s_n\}$ be a sequence in K. We argue:

(*) $\exists x \in K$ so that $\forall \varepsilon > 0$ $\{n \in \mathbb{N} \mid s_n \in B_\varepsilon(x)\}$ is infinite.

(*) $\Rightarrow$ there is a subsequence that converges to x:
take $\varepsilon = 1, 1/2, 1/3 \ldots$ and find $n_1 < n_2 < \cdots < n_k < \cdots$ s.t
$s_{n_k} \in B_{1/k}(x)$

Suppose (*) does not hold. Then $\forall x \in K \ \exists \varepsilon(x)$ st
$\{n \in \mathbb{N} \mid s_n \in B_{\varepsilon(x)}(x)\}$ is finite.

The collection $\{B_{\varepsilon(x)}(x)\}_{x \in K}$ is an open cover of K.
Since K is compact, it has a finite subcover:
$K \subseteq B_{\varepsilon(x_1)}(x_1) \cup \cdots \cup B_{\varepsilon(x_k)}(x_k)$.

But $\{n \in \mathbb{N} \mid s_n \in B_{\varepsilon(x_1)}(x_1) \cup \cdots \cup B_{\varepsilon(x_k)}(x_k)\}$
is finite. Contradiction.

Definition A subset K of a metric space is totally bounded if $\forall \varepsilon > 0$
$\exists x_1 \ldots x_n \in K$ st $K \subseteq B_\varepsilon(x_1) \cup \cdots \cup B_\varepsilon(x_n)$, ie
$\forall \varepsilon$ K can be covered by finitely many balls of radius ε.

Lemma 11.2 Suppose (E,d) is a metric space, $K \subseteq E$ sequentially compact.
Then (K,d) is complete and K is totally bounded.

Proof Suppose $\{s_n\} \subseteq K$ is Cauchy. Since K is sequentially compact
$\{s_n\}$ has a convergent subsequence with limit L in K.
Since $\{s_n\}$ is Cauchy, $s_n \to L$.

Suppose K is not totally bounded. Then $\exists \varepsilon > 0$ st K cannot
be covered by finitely many ε-balls $\Rightarrow \exists x_1$ st $K \setminus B_\varepsilon(x_1) \neq \emptyset$.
$\exists x_2 \in K \setminus B_\varepsilon(x_1)$ st $K \setminus (B_\varepsilon(x_2) \cup B_\varepsilon(x_1)) \neq \emptyset$

$\vdots$

$\exists x_n \in K \setminus (B_\varepsilon(x_1) \cup \cdots \cup B_\varepsilon(x_{n-1}))$ st $K \setminus (B_\varepsilon(x_1) \cup \cdots \cup B_\varepsilon(x_n)) \neq \emptyset$

We get a sequence $\{x_n\}$ in K with $d(x_n, x_m) \geq \varepsilon$ for $n \neq m$.
$\Rightarrow \{x_n\}$ has no Cauchy subsequence and hence no convergent subsequence

Contradiction. ☐

Lemma 11.3 (E, d) metric space, $K \subseteq E$ complete and totally bounded. Then K is compact.

Proof Suppose $\exists$ an open cover $\{U_\alpha\}_{\alpha \in A}$ of K with no finite subcover. Since K is totally bounded, K can be covered by finitely many open balls of radius 1. $\Rightarrow \exists x_0 \in K$ s.t.

$B_1(x_0)$ cannot be covered by finitely many U_α's.

There is a finite cover of K by balls of radius $1/2$.

Consider a subset of these balls that cover $B_1(x_0)$

Then one of these balls, say $B_{1/2}(x_1)$ cannot be covered by finitely many U_α's.

Proceeding this way we get a sequence $x_0, x_1, \dots x_n \dots$

s.t. $B_{1/2^n}(x_n) \cap B_{1/2^{n+1}}(x_{n+1}) \neq \emptyset$

(and then $d(x_n, x_{n+1}) < \frac{1}{2^n} + \frac{1}{2^{n+1}} < \frac{2}{2^n} = \frac{1}{2^{n-1}}$

and $B_{1/2^n}(x_n)$ cannot be covered by finitely many U_α's

Since $d(x_n, x_{n+k}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \dots + d(x_{n+k-1}, x_{n+k})$

$$< \frac{1}{2^{n-1}}\left(1 + \frac{1}{2} + \dots + \frac{1}{2^k}\right) < \frac{2}{2^{n-1}} = \frac{1}{2^{n-2}}$$

$\{x_n\}$ is Cauchy. Since K is complete, $x_n \to y$ for some $y \in K$.

Since $\{U_\alpha\}_{\alpha \in A}$ covers K, $\exists \alpha_0$ s.t. $y \in U_{\alpha_0}$.

Since U_{α_0} is open $\exists r > 0$ s.t. $B_r(y) \subseteq U_{\alpha_0}$

Since $x_n \to y$ $\exists n$ s.t. $x_n \in B_{r/2}(y)$ and $\frac{1}{2^n} < \frac{r}{2}$.

Then $B_{1/2^n}(x_n) \subseteq B_r(y) \subseteq U_{\alpha_0}$.

Contradiction, since $B_{1/2^n}(x_n)$ cannot be covered by one U_α. ☐

Summary For a subset of a metric space,

compact $\Rightarrow$ sequentially compact $\Rightarrow$ complete and totally bounded $\Rightarrow$ compact.

Thus all three are equivalent.

<u>Heine-Borel Theorem</u> A subset K of $\mathbb{R}^n$ is compact $\Leftrightarrow$ K is closed and bounded.

<u>Proof</u> ($\Rightarrow$) true in any metric space.

($\Leftarrow$) Suppose K is closed and bounded. By Ex 11.0, K is sequentially compact. By 11.2 K is complete and totally bounded. By 11.3 K is compact. $\square$

<u>Ex.</u> $\mathbb{R}$, $d(x,y) = \min(1, |x-y|)$

$(\mathbb{R}, d)$ is bounded but it's <u>not</u> totally bounded since it's complete but <u>not</u> compact. (in fact [illegible]

$$B^d_{1/2}(x) = (x-1/2, x+1/2) \quad \forall x$$

and $\mathbb{R}$ cannot be covered by finitely many balls of radius $1/2$.

Thus only in $(\mathbb{R}^n, d_2)$ (or d_1 or d_∞)

bounded $\Rightarrow$ totally bounded.

<u>Next time</u> continuous functions

(we'll discuss the last bit of Chapter 3, connected sets, later)