MATH 424 Eugene Lerman 02/12/2024

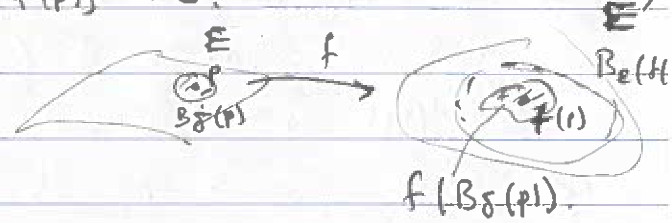
12.1

Definition Let (E,d), (E',d') be two metric spaces. A function $f: E \to E'$ is <u>continuous</u> at $p \in E$ if

$\forall \varepsilon > 0 \ \exists \delta > 0$ so that $\forall x \in E$

$$d(x,p) < \delta \Rightarrow d'(f(x), f(p)) < \varepsilon.$$

i.e. $f(B^d_\delta(p)) \subseteq B^{d'}_\varepsilon(f(p))$



A function $f: E \to E'$ is <u>continuous</u> if it is continuous at every point p of E.

Ex (E,d) metric space, $q \in E$ a point, $f: E \to \mathbb{R}$, $f(p) = d(q,p)$

Then f is continuous at every $p \in E$:

$$|f(x) - f(p)| = |d(x,q) - d(p,q)| \leq d(x,p)$$
↑ triangle inequality

So $\forall \varepsilon > 0$, $d(x,p) < \varepsilon \Rightarrow |f(x) - f(p)| < \varepsilon$

(i.e., we may take $\delta = \varepsilon$).

Ex $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} 1 & x \text{ rational} \\ 0 & x \text{ irrational} \end{cases}$ is not continuous at <u>any</u> $p \in \mathbb{R}$.

Reason $\forall p \in \mathbb{R}$ $\forall \delta > 0$ $B_\delta(p) = (p-\delta, p+\delta)$ contains both rationals and irrationals. So if p is rational (and $f(p) = 1$)

$\forall \delta > 0 \ \exists x \in B_\delta(p)$ s.t. $f(x) = 0$

and Then, for any $\varepsilon < 1$ (say $\varepsilon = 1/2$)

no matter which δ we choose $|x - p| < \delta \not\Rightarrow |f(x) - 1| < \frac{1}{2}$

Similar problem if $f(p) = 0$.

Theorem 12.1 $f: (E,d) \to (E',d')$ is continuous

$\Leftrightarrow \forall U \subseteq E'$, open, $f^{-1}(U)$ is open.

Proof ($\Rightarrow$) Suppose f is continuous, $\mathcal{U} \subseteq E'$ open.

$\forall p \in f^{-1}(\mathcal{U})$ $f(p) \in \mathcal{U}$. $\Rightarrow \exists \varepsilon > 0$ st $B_\varepsilon(f(p)) \subseteq \mathcal{U}$

Since f is continuous at p $\exists \delta > 0$ st $f(B_\delta(p)) \subseteq B_\varepsilon(f(p))$

$\Rightarrow$ $f(B_\delta(p)) \subseteq \mathcal{U}$

$\Rightarrow$ $B_\delta(p) \subseteq f^{-1}(\mathcal{U})$. Since $p \in f^{-1}(\mathcal{U})$ is arbitrary

$f^{-1}(\mathcal{U})$ is open.

($\Leftarrow$) Suppose $\forall$ open $\mathcal{U} \subseteq E'$, $f^{-1}(\mathcal{U})$ is open in E.

Given $p \in E$ and $\varepsilon > 0$, $B_\varepsilon(f(p))$ is open in E'

By assumption $f^{-1}(B_\varepsilon(f(p)))$ is open in E.

Since $p \in f^{-1}(f(p)) \subseteq f^{-1}(B_\varepsilon(f(p)))$

and since $f^{-1}(B_\varepsilon(f(p)))$ is open $\exists \delta > 0$ st

$B_\delta(p) \subseteq f^{-1}(B_\varepsilon(f(p)))$

$\Rightarrow$ $f(B_\delta(p)) \subseteq B_\varepsilon(f(p))$ and f is continuous at p. $\square$

Corollary 12.2 $f: (E,d) \to (E',d')$ is continuous iff

$\forall$ closed set $C \subseteq E'$, $f^{-1}(C)$ is closed.

Definition A map/function $f: (X, \mathcal{T}) \to (X', \mathcal{T}')$ between two topological spaces is continuous if $\forall\, \mathcal{U} \in \mathcal{T}'$ $f^{-1}(\mathcal{U}) \in \mathcal{T}$, ie. preimages of open sets are open.

Remark Thm 12.1 says that the notion of continuity of a map depends only on the topologies:

if d_1, d_2 are two metrics on E s.t. $\mathcal{T}_{d_1} = \mathcal{T}_{d_2}$

d_1', d_2' —— // —— on E' s.t. $\mathcal{T}_{d_1'} = \mathcal{T}_{d_2'}$

Then $f: (E, d_1) \to (E', d_1')$ is continuous

$\Leftrightarrow$ $f: (E, d_2) \to (E', d_2')$ is continuous.

Theorem 12.3 Composite of two continuous maps is continuous: if $f: (X, T_X) \to (Y, T_Y)$, $g: (Y, T_Y) \to (Z, T_Z)$ are continuous then $g \circ f: (X, T_X) \to (Z, T_Z)$ is continuous.

Proof Suppose $W \subseteq Z$ is open. Then $g^{-1}(W) \subseteq Y$ is open
$\Rightarrow$ $f^{-1}(g^{-1}(W))$ is open in X
But $f^{-1}(g^{-1}(W)) = (g \circ f)^{-1}(W)$.
$\Rightarrow$ $g \circ f$ is continuous. $\square$

Theorem 12.4 Images of compact sets under continuous maps are compact: if $f: X \to Y$ is continuous, and $K \subseteq X$ is compact, then $f(K) \subseteq Y$ is compact.

Proof Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of $f(K)$.
Then $\{f^{-1}(U_\alpha)\}_{\alpha \in A}$ is an open cover of K.
Since K is compact $\exists \alpha_1 \ldots \alpha_n$ st $K \subseteq f^{-1}(U_{\alpha_1}) \cup \ldots \cup f^{-1}(U_{\alpha_n})$
$\Rightarrow f(K) \subseteq U_{\alpha_1} \cup \ldots \cup U_{\alpha_n}$ and $f(K)$ is compact.

Corollary 12.5 (E, d) metric space, X topological space $K \subseteq X$ compact and $f: X \to E$ continuous.
Then $f(K)$ is complete, totally bounded, sequentially compact and closed.

Corollary 12.6 Suppose X is a top space, $f: X \to \mathbb{R}$ continuous and $K \subseteq X$ compact. Then $\exists x_1, x_2 \in X$ st
$\forall x \in X$ $f(x_1) \leq f(x) \leq f(x_2)$,
ie f achieves max and min on X.

Proof $f(K)$ is closed and bounded in $\mathbb{R}$ hence

$\exists x_1, x_2 \in K$ s.t $f(x_1) = \inf(f(K))$, $f(x_2) = \sup(f(K))$. $\square$

<u>Definition</u> Let X be a top space, $S \subset X$ a subset. $x \in X$ is a <u>cluster point</u> of S if $\forall$ open set $\mathcal{U}$ with $x \in \mathcal{U}$, $(\mathcal{U} \setminus \{x\}) \cap S$ is nonempty.

If E is a metric space, x is a cluster point of S
$\Leftrightarrow$ $\exists$ a sequence $\{s_n\} \subseteq S \setminus \{x\}$ st $s_n \to x$.

<u>Ex</u> $S := \{0\} \cup [1,2] \subseteq \mathbb{R}$, $d(x,y) = |x-y|$

1 is a cluster point of S; 0 is not a cluster point of S.
<u>Any</u> $x \in [1,2]$ is a cluster point of S.

<u>Definition</u> Let (E,d), (E',d') be two metric spaces, $A \subseteq E$, $f: A \to E'$ a function, p a cluster point of A. Then

$$\lim_{x \to p} f(x) = q \qquad \text{(the limit of } f \text{ at } p \text{ is } q\text{)}$$

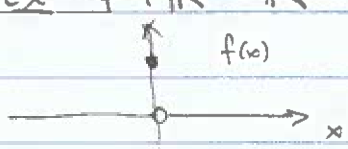
if $\forall \varepsilon > 0$ $\exists \delta > 0$ so that

if $x \in A \cap B_\delta(p)$, $x \neq p$, then $d'(f(x), q) < \varepsilon$.

<u>Remarks</u>

1) We're not assuming that f is defined at p
2) Even if f is defined at p we're not assuming $f(p) = q$.

<u>Ex</u> $f: \mathbb{R} \to \mathbb{R}$ $f(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$ $\lim_{x \to 0} f(x) = 0 \neq 1 = f(0)$



<u>Lemma</u> Let p be a cluster point of a metric space E, E' metric space. $f: E \to E'$ is continuous at p $\Leftrightarrow$ $\lim_{x \to p} f(x) = f(p)$.

<u>Proof</u> easy (?) exercise