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13.1

Recall If (E,d) is a metric space, $S \subseteq E$ a subset Then $x \in E$ is a <u>cluster point</u> of S if $\exists$ a sequence $\{s_n\} \subseteq S \setminus \{x\}$ s.t $s_n \to x$.

In general: x is a cluster point of S if $\forall$ open set U with $x \in U$, $(U \setminus \{x\}) \cap S \neq \emptyset$

Definition (E,d), (E',d') two metric spaces, $A \subseteq E$, $f: A \to E'$ p a cluster point of A. Then

$$\lim_{x \to p} f(x) = q \quad [\text{the limit of } f(x) \text{ as } x \to p \text{ is } q]$$

if $\forall \varepsilon > 0$ $\exists \delta > 0$ so that

$$(x \in A \cap B_\delta(p),\ x \neq p) \Rightarrow d'(f(x), q) < \varepsilon.$$

Remarks (1) We are <u>not</u> assuming $p \in A$ (so $f(p)$ may not be defined)

(2) Even if $p \in A$ we are not requiring that $f(p) = \lim_{x \to p} f(x)$.

Ex $f: \mathbb{R} \to \mathbb{R}$ $\quad f(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$ $\quad \lim_{x \to 0} f(x) = 0 \neq 1 = f(x)$.

Lemma Suppose E, E' metric spaces, p cluster point of E. Then $f: E \to E'$ is continuous at $p \iff \lim_{x \to p} f(x) = f(p)$.

Proof Exercise.

Note If p is <u>not</u> a cluster point of E $\exists\, r > 0$ s.t $(B_r(p) \setminus \{p\}) \cap E = \emptyset$ i.e. $B_r(p) = \{p\}$. And then any $f: E \to E'$ is continuous at p.

Theorem 13.1 E, E' two metric spaces. $f: E \to E'$ is continuous at $p \in E$ $\iff$ $\forall$ sequence $\{s_n\}$ in E

$$s_n \to p \text{ in } E \implies f(s_n) \to f(p) \text{ in } E'.$$

Proof $(\Rightarrow)$ Suppose $s_n \to p$ and f is continuous. Given $\varepsilon > 0$ $\exists \delta > 0$ s.t $d(x,p) < \delta \Rightarrow d'(f(x), f(p)) < \varepsilon$.

Since $s_n \to p$ $\exists N$ s.t $n \geq N \Rightarrow d(s_n, p) < \delta$. $\Rightarrow d'(f(s_n), f(p)) < \varepsilon$

$\therefore f(s_n) \to f(p)$.

$(\Leftarrow)$ Suppose f is <u>not</u> continuous at p. We construct $\{s_n\}$

with $s_n \to p$ and $f(s_n) \not\to f(p)$.

Since f is <u>not</u> continuous at p $\exists\, \varepsilon_0 > 0$ so that $\forall \delta > 0$ $\exists\, x_\delta \in B_\delta(p)$ with $f(x_\delta) \notin B_{\varepsilon_0}(f(p))$.

Let $s_n = x_{1/n}$. Then $s_n \in B_{1/n}(p)$ (hence $s_n \to p$) and $f(s_n) \notin B_{\varepsilon_0}(f(p))$ (hence $f(s_n) \not\to f(p)$). ◻

Note that since $f(s_n) \neq f(p)$, $s_n \neq p$ $\forall n$.

<u>Remark</u> By definition of the limit of f at p

$$\lim_{x \to p} f(x) = q \iff \forall \text{ sequence } \{s_n\} \in E \setminus \{p\}$$
$$s_n \to p \Rightarrow f(s_n) \to q.$$

<u>Theorem 13.2</u> Suppose $f, g : (E, d) \to \mathbb{R}$ are continuous at $p \in E$. Then $f+g$, $f \cdot g$ are continuous at p. If $g(p) \neq 0$ then f/g is continuous at p.

<u>Proof</u> Suppose $s_n \to p$. Then $f(s_n) \to f(p)$, $g(s_n) \to g(p)$ by continuity of f, g at p. $\Rightarrow$ $(f+g)(s_n) = f(s_n) + g(s_n) \to f(p) + g(p)$
$$= (f+g)(p)$$
by theorems about limits of sequences. $\Rightarrow$ $f+g$ is continuous at p.

The rest of the proof is similar. ◻

<u>Theorem 13.3</u> Suppose $f = (f_1 \ldots f_n) : E \to \mathbb{R}^n$ is a function, $p \in E$. Then f is continuous at $p \iff f_1, \ldots f_n$ are all continuous at p.

<u>Proof</u> Recall that a sequence $t_k = (t_k^{(1)}, \ldots t_k^{(n)})$ in $\mathbb{R}^n$ converges to $(q_1 \ldots q_n) \iff t_k^{(i)} \to q_i \;\forall i$

◻

Uniform continuity

Recall: $f:(E,d)\to(E',d')$ is continuous if $\forall p\in E$, $\forall \varepsilon>0$ $\exists\, \delta=\delta_{\varepsilon,p}>0$ s.t. $d(x,p)<\delta \Rightarrow d'(f(x),f(p))<\varepsilon$.

Definition $f:(E,d)\to(E',d')$ is uniformly continuous if $\forall\varepsilon>0$ $\exists\,\delta=\delta_\varepsilon>0$

$$d(x,p)<\delta \Rightarrow d'(f(x),f(p))<\varepsilon$$

(this is the same δ for all x and p).

Ex $f(x)=x^2$, $f:[0,1]\to\mathbb{R}$ is uniformly continuous:

$$|f(x)-f(y)| = |x^2-y^2| = |x-y|\,|x+y| \le |x-y|(|x|+|y|) \le 2|x-y|.$$

Thus given $\varepsilon>0$, $|x-y|<\varepsilon/2 \Rightarrow |f(x)-f(y)|<\varepsilon$.

$f(x)=x^2$, $f:(0,\infty)\to\mathbb{R}$ is not uniformly continuous:

$|f(x)-f(y)| = |x-y|\,|x+y| \ge 2\min\{x,y\}\,|x-y|$.

Therefore $\forall\,\delta$ if $x,y>1/\delta$ and $|x-y|=\delta/2<\delta$ we have $|f(x)-f(y)|>1$.

Lemma 13.4 Suppose $f:E\to E'$ is uniformly continuous. Then for any Cauchy sequence $\{x_n\}$ in E, $\{f(x_n)\}$ is Cauchy in E'.

Proof Since f is uniformly continuous $\forall\varepsilon>0$ $\exists\,\delta_\varepsilon>0$ s.t.

$$d(x,y)<\delta_\varepsilon \Rightarrow d'(f(x),f(y))<\varepsilon. \quad \forall x,y\in E$$

Since $\{x_n\}$ is Cauchy $\exists N$ st $n,m\ge N \Rightarrow d(x_n,x_m)<\delta_\varepsilon$

Hence for $n,m\ge N$, $d'(f(x_n),f(x_m))<\varepsilon$

$\therefore$ $\{f(x_n)\}$ is Cauchy $\square$

Ex $f(x)=\sin 1/x$ is not uniformly continuous on $(0,1)$

[it is continuous; but we haven't really discussed $\sin(u)$]

Consider $s_n = \frac{1}{\pi/2 + \pi n} \in (0,1)$. $s_n \to 0$ hence is Cauchy in $(0,1)$.
But $\sin(\frac{1}{s_n}) = \sin(\frac{\pi}{2} + \pi n) = (-1)^n$ is not Cauchy.

<u>Theorem 13.5</u> Suppose $f: E \to E'$ is continuous and E is compact. Then f is uniformly continuous.

<u>Proof</u> Given $\varepsilon > 0$ we want to show: $\exists \delta$ s.t. $d(x,y) < \delta \Rightarrow d(f(x), f(y)) < \varepsilon$.
Since f is continuous $\forall x \in E$ $\exists \delta_x > 0$ s.t.

$$(*) \quad d(x,y) < \delta_x \Rightarrow d(f(x), f(y)) < \varepsilon/2.$$

$\{B_{\delta_x/2}(x)\}_{x \in E}$ is an open cover of E. Since E is compact,
$\exists$ $x_1 \ldots x_n$ st $E = B_{\delta_{x_1}/2}(x_1) \cup \cdots \cup B_{\delta_{x_n}/2}(x_n)$.
Let $\delta = \min\{\delta_{x_1}/2, \ldots \delta_{x_n}/2\}$.
Suppose $d(p,q) < \delta$. Then $q \in B_{\delta_{x_i}/2}(x_i)$ for some i.

$\Rightarrow$ $d(p, x_i) \leq d(p,q) + d(q, x_i) < \delta + \delta_{x_i}/2 \leq \delta_{x_i}/2 + \delta_{x_i}/2$

$\Rightarrow$ $p \in B_{\delta_{x_i}}(x_i)$

$\Rightarrow$ Since $p, q \in B_{\delta_{x_i}}(x_i)$, $d'(f(p), f(x_i)), d'(f(q), f(x_i)) < \varepsilon/2$.
by $(*)$.

$$\Rightarrow d'(f(p), f(q)) \leq d'(f(p), f(x_i)) + d'(f(x_i), f(q)) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

$\square$