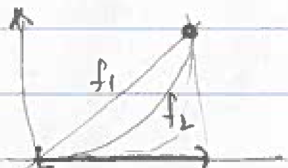
MATH 424 Eugene Lerman

Sequences of functions and their convergence

Definition Let $\{f_n : (E,d) \to (E',d')\}_{n=1}^{\infty}$ be a sequence of functions between two metric spaces. The sequence converges to $f : E \to E'$ pointwise if $\forall p \in E$, $f_n(p) \to f(p)$. We write $f = \lim_{n\to\infty} f_n$.

Ex $E = [0,1] = E'$, $f_n(x) = x^n$.

$$\lim_{n\to\infty} x^n = \begin{cases} 0 & 0 \le x < 1 \\ 1 & x = 1 \end{cases}$$



Note 1. $f(x) = \lim_{n\to\infty} x^n$ is not continuous even though each $f_n(x) = x^n$ is continuous!

2. $\{f_n(x) = x^n : [0,2] \to \mathbb{R}\}$ does not converge since, for example $\{2^n\}_{n=1}^{\infty}$ does not converge.

Definition $\{f_n : (E,d) \to (E',d')\}$ a sequence of functions, $A \subseteq E$ subset. $f_n \to f$ uniformly on A if $\forall \varepsilon > 0$ $\exists N$ so that

$$n \ge N \Rightarrow d'(f_n(p), f(p)) < \varepsilon \quad \forall p \in A.$$

Equivalently: $\forall \varepsilon > 0$ $\exists N$ s.t.

$$n \ge N \Rightarrow \sup\{d'(f_n(p), f(p)) \mid p \in A\} < \varepsilon.$$

ie $\lim_{n\to\infty} \sup\{d'(f_n(p), f(p)) \mid p \in A\} = 0$

Ex $f_n(x) = x^n : [0,1] \to [0,1]$ converges uniformly on $A = [0,a]$ for all $a < 1$; it does not converge uniformly on $[0,1)$.

Reason:

$$\sup\{|x^n - 0| \mid 0 \le x \le a\} = a^n \xrightarrow[n\to\infty]{} 0$$

$$\sup\{|x^n - 0| \mid 0 \le x < 1\} = 1 \underset{n\to\infty}{\not\to} 0$$

Ex $f_n(x) = \frac{nx}{1+n^2x^2} \to 0$ pointwise on $\mathbb{R}$ but not uniformly:

$f_n(0) = 0 \to 0$; for $x \neq 0$, $|f_n(x)| \leq \frac{n|x|}{n^2|x|^2} = \frac{1}{n|x|} \underset{n\to\infty}{\to} 0$

Since $f_n(\frac{1}{n}) = \frac{1}{1+n^2\frac{1}{n^2}} = 1/2$

$\sup\{|f_n(x) - 0| \mid x \in \mathbb{R}\} \geq \frac{1}{2} \not\to 0.$

so the convergence to 0 is not uniform.

Ex $f_n(x) = \frac{x}{n^2+x^2} \to 0$ uniformly on $\mathbb{R}$.

Check

$0 \leq (n - |x|)^2 = n^2 - 2n|x| + x^2$, $\Rightarrow$ $n^2 + x^2 \geq 2n|x|$.

$\Rightarrow$ for $x \neq 0$, $|f_n(x)| = \frac{|x|}{n^2+x^2} \leq \frac{|x|}{2n|x|} = \frac{1}{2n} \to 0$

(for $x = 0$ $|f_n(x)| = 0) \leq \frac{1}{2n}$

$\Rightarrow$ $\sup\{|f_n(x) - 0| \mid x \in \mathbb{R}\} \leq \frac{1}{2n} \to 0$.

Uniform Cauchy criterion

Definition A sequence of functions $\{f_n : E \to E'\}_{n\in\mathbb{N}}$ is uniformly Cauchy on $A \subseteq E$ if

$\forall \varepsilon > 0$ $\exists N$ s.t. $n, m \geq N \Rightarrow \sup\{d'(f_n(p), f_m(p)) \mid p \in A\} < \varepsilon$

Theorem 14.1 $\{f_n : E \to E'\}_{n\in\mathbb{N}}$ sequence of functions, E' complete.

Then $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly on A $\Leftrightarrow$

$\{f_n\}_{n\in\mathbb{N}}$ is uniformly Cauchy on A.

Proof ($\Rightarrow$) Suppose $f_n \to f$ uniformly on A. Then $\forall \varepsilon > 0$ $\exists N$ s.t.

$\forall n \geq N \Rightarrow \sup\{d'(f_n(p), f(p)) \mid p \in A\} < \varepsilon/3$

$\Rightarrow \forall n, m \geq N$ $\forall p$ $d'(f_n(p), f_m(p)) \leq d'(f_n(p), f(p)) + d'(f_m(p), f(p))$

$< \varepsilon/3 + \varepsilon/3$

$\Rightarrow \sup\{d'(f_n(p), f_m(p)) \mid p \in A\} \leq \frac{2\varepsilon}{3} < \varepsilon.$

($\Leftarrow$) Suppose $\{f_n\}$ is uniformly Cauchy on A. Then $\forall p \in A$ $\{f_n(p)\}_{n \in \mathbb{N}}$ is Cauchy in E'. Since E' is complete $\{f_n(p)\}$ converges. Define $f : E \to E'$ by

$$f(p) = \lim_{n\to\infty} f_n(p).$$

We now argue that $f_n \to f$ uniformly on A.

(*) Recall: $\forall x \in E'$, $h : E' \to \mathbb{R}$, $h(y) := d'(x,y)$ is continuous.

Since $\{f_n\}$ is uniformly Cauchy on A, given $\varepsilon > 0$ $\exists N$ st $\forall m, n \geq N$, $\sup\{d'(f_n(p), f_m(p)) \mid p \in A\} < \varepsilon$

Fix $n \geq N$, $p \in A$. Then

$$d'(f_n(p), f(p)) \underset{(*)}{=} \lim_{m\to\infty} d'(f_n(p), f_m(p)) \leq \sup_{m > n} d'(f_n(p), f_m(p)) < \varepsilon$$

$\therefore$ $\forall n \geq N$ $\sup\{d'(f_n(p), f(p)) \mid p \in A\} < \varepsilon$, i.e. $f_n \to f$ uniformly on A.

$\square$

Theorem 14.2 Uniform limit of continuous functions is continuous, i.e. if $\{f_n : E \to E'\}_{n \in \mathbb{N}}$ converges uniformly (on E) to f then f is continuous.

Proof Fix $p \in E$. For any $x \in E$, for any $n \in \mathbb{N}$

$$d'(f(p), f(x)) \leq d'(f(p), f_n(p)) + d'(f_n(p), f_n(x)) + d'(f_n(x), f(x)).$$

Given $\varepsilon > 0$ we want to show: $\exists \delta > 0$ so that

$$d(x,p) < \delta \implies d'(f(p), f(x)) < \varepsilon$$

This would prove continuity at p, hence everywhere.

Since $f_n \to f$ uniformly $\exists N$ st for $n \geq N$ $d'(f_n(y), f(y)) < \varepsilon/3$ $\forall y$

Since f_N is continuous at p $\exists \delta$ st.

$$d(x,p) < \delta \Rightarrow d'(f_N(x), f_N(p)) < \varepsilon/3$$

Then for any x with $d(x,p) < \delta$

$$d'(f(x), f(p)) \le d'(f(x), f_N(x)) + d'(f_N(x), f_N(p)) + d'(f_N(p), f(p))$$
$$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. \qquad \square$$

<u>Definition</u> Let (E,d), (E',d') be two metric spaces.
A function $f: E \to E'$ is bounded if its image
$$f(E) := \{ f(x) \mid x \in E \}$$ is bounded in E'.

<u>Notation</u>
$$C(E,E') = \{ f: E \to E' \mid f \text{ is bounded and continuous} \}.$$

<u>Exercise</u> 1) If $f, g : E \to E'$ are bounded then the set
$$\{ d'(f(x), g(x)) \mid x \in E \}$$ is bounded.

2) Define $D: C(E,E') \times C(E,E') \to [0,\infty)$ by
$$D(f,g) = \sup \{ d'(f(x), g(x)) \mid x \in E \}.$$
Check that $(C(E,E'), D)$ is a metric space.

Why did we define all of this?
$f_n \to f$ in $C(E,E') \iff f_n \to f$ uniformly on E.
$\{f_n\}$ is Cauchy in $C(E,E') \iff \{f_n\}$ is uniformly Cauchy on E.

<u>Theorem 14.3</u> If E' is a complete metric space then $(C(E,E'), D)$ is complete.

<u>Proof</u> $\{f_n\}$ is Cauchy in $(C(E,E'), D) \Rightarrow \{f_n\}$ is uniformly Cauchy on E. $\Rightarrow \{f_n\}$ converges uniformly to a function f on E.
Since each f_n is continuous and convergence is uniform, f is continuous.
Since each f_n is bounded, f has to be bounded as well
[prove this; it should not be hard].