Recall (E,d), (E',d') metric spaces

$C(E,E') = \{f : E \to E' \mid f \text{ is bounded and continuous}\}$

Then (exercise)

$D : C(E,E') \times C(E,E') \to [0,\infty)$,

$D(f,g) = \sup\{d'(f(x),g(x)) \mid x \in E\}$ is a well-defined metric.

and $f_n \to f$ in $(C(E,E'), D) \iff f_n \to f$ uniformly on E

$\{f_n\}_{n\in\mathbb{N}}$ is Cauchy in $C(E,E') \iff \{f_n\}$ is uniformly Cauchy

Thm 14.3 (left over from a week ago)

If E' is a complete metric space then $C(E,E')$ is complete.

Proof If a sequence $\{f_n\}$ is Cauchy in $C(E,E')$ then it is uniformly Cauchy on E. By 14.1, $\{f_n\}$ converges uniformly to a function $f : E \to E'$.

Since convergence is uniform, $f = \lim f_n$ is continuous.

Moreover since each f_n is bounded, and $f_n \to f$ uniformly f is bounded. (HW)

$\therefore \quad f = \lim f_n \in C(E,E')$ $\square$

Definition A subset Y of a topological space X is connected if $\forall\, U, V \subseteq X$, open, with $Y \subset U \cup V$ and $(Y\cap U)\cap(Y\cap V) = \emptyset$ either $Y \subseteq U$ or $Y \subseteq V$.

(In particular a space X is connected if $\forall$ open cover $\{U, V\}$ of X with $U \cap V = \emptyset$ either $U = \emptyset$ or $V = \emptyset$)

Non Ex $Y = [0,1/2) \cup (1/2, 1] \subseteq \mathbb{R}$ is not connected:

Let $U = (-\infty, 1/2)$, $V = (1/2, +\infty)$. Then $U \cap V = \emptyset$

$Y \subseteq U \cup V$, and $Y \cap U = [0, 1/2) \neq \emptyset$, $Y \cap V = (1/2, 1] \neq \emptyset$.

Theorem 15.1 $[0,1]$ (standard topology) is connected.

Proof Suppose $[0,1] = U \cup V$, U, V open $U \cap V = \emptyset$

May assume: $0 \in U$. We argue: $V = \emptyset$.

Let $S = \{x \in [0,1] \mid [0,x] \subseteq U\}$

Since U is open, $0 \in U$, $\exists r > 0$ s.t. $B_r(0) = [0,r) \subseteq U$.

Then $[0, r/2] \subseteq [0,r) \subseteq U$. $\Rightarrow r/2 \in S \Rightarrow S \neq \emptyset$.

Since $S \subseteq [0,1]$, S is bounded above by 1.

Let $c := \sup S$. Then $c \leq 1$. And, since $c > r/2$, $c > 0$.

<u>Claim 1</u> $[0,c) \subseteq U$

<u>Proof</u> If $y \in [0,c)$ then $y < c = \sup S$. $\Rightarrow z \in S$ s.t. $y \leq z < \sup S = c$

$\Rightarrow [0,z] \subseteq U$. $\Rightarrow [0,y] \subset [0,z] \subseteq U$. $\Rightarrow y \in U$. $\therefore [0,c) \subseteq U$.

<u>Claim 2</u> $c \in U$.

<u>Proof</u> If $c \notin U$, $c \in V$ (since $[0,1] = U \cup V$)

Since V is open, $\exists \delta > 0$ s.t. $(c-\delta, c+\delta) \subseteq V$. $\Rightarrow c - \delta/2 \in V$

But $c - \delta/2 < c$ hence $c - \delta/2 \in [0,c) \subseteq U$. Contradiction.

$\therefore$ $[0,c] = [0,c) \cup \{c\} \subseteq U$.

We now argue $c = 1$ (hence $[0,1] \subseteq U$ and $V = \emptyset$):

<u>If</u> $c \neq 1$, then $c < 1$, $\Rightarrow c \in (0,1)$.

Since $(0,1)$, U are open, $(0,1) \cap U$ is open. And since

$c \in (0,1) \cap U$ $\exists \varepsilon > 0$ s.t. $(c-\varepsilon, c+\varepsilon) \in (0,1) \cap U$.

$\Rightarrow [0, c+\varepsilon) = [0,c] \cup (c-\varepsilon, c+\varepsilon) \subseteq U$

$\Rightarrow [0, c+\varepsilon/2] \subseteq U$. $\Rightarrow c + \varepsilon/2 \in S$

Contradiction since $c = \sup S$.

$\therefore$ $[0,1]$ is connected.

<u>Theorem 15.2</u> Suppose $f: X \to Y$ is continuous, X connected.

Then $f(X) \subseteq Y$ is connected.

<u>Proof</u> Suppose $U, V \subseteq Y$ are open,

$f(X) \subset U \cup V$. and $(f(X) \cap U) \cap (f(X) \cap V) = \emptyset$.

We argue $f(X) \subseteq U$ or $f(X) \subseteq V$. Since $f(X) \subseteq U \cup V$, $X = f^{-1}(U) \cup f^{-1}(V)$

Since f is continuous, $f^{-1}(V)$, $f^{-1}(U)$ are open.
If $x \in f^{-1}(V) \cap f^{-1}(U)$ then $f(x) \in V$ and $f(x) \in U$
But $V \cap U = \emptyset$, so $f^{-1}(U) \cap f^{-1}(V) = \emptyset$.
Since X is connected either $f^{-1}(U) = \emptyset$ or $f^{-1}(V) = \emptyset$.
Hence either $f(X) \subseteq V$ (if $f^{-1}(U) = \emptyset$) or $f(X) \subseteq U$ (if $f^{-1}(V) = \emptyset$)
$\therefore$ $f(X)$ is connected

Corollary 15.3 $\forall a, b \in \mathbb{R}$, $a < b$, $[a,b]$ is connected
Proof $f : [0,1] \to [a,b]$ $f(t) = ta + (1-t)b$
is continuous and $f([0,1]) = [a,b]$.
Since $[0,1]$ is connected, $[a,b]$ is connected by 15.2. $\square$

Definition A space X is path-connected if $\forall p, q \in X$
$\exists$ continuous function $\gamma : [0,1] \to X$ with $\gamma(0) = p$, $\gamma(1) = q$.
γ is called a path from p to q.

Ex $\mathbb{R}^n$ is path connected: $\forall p, q \in \mathbb{R}^n$
$\gamma(t) = tp + (1-t)q$ is a continuous path.

$B_r(0) \subseteq \mathbb{R}^n$ is path connected: (for any norm $\|\cdot\|$ on $\mathbb{R}^n$)
$\forall p, q \in B_r(0)$ consider $\gamma(t) = tp + (1-t)q$, $t \in [0,1]$
Then $\|\gamma(t) - 0\| = \| tp + (1-t)q\| \leq \|tp\| + \|(1-t)q\|$
$= |t|\,\|p\| + |1-t|\,\|q\|$
$\leq t \cdot 1 + (1-t) \cdot 1 = 1$
$\Rightarrow \gamma(t) \in B_r(0)$ $\forall t$.

Theorem 15.4 If X is path connected then X is connected.
Proof Suppose $U, V \subseteq X$ are disjoint open sets with $U \cup V = X$.

If U, V are both nonempty $\exists\, p \in U$, $q \in V$. Since X is path connected $\exists\, \gamma : [0,1] \to X$, continuous, with $\gamma(0) = p \in U$, $\gamma(1) = q \in V$.
Since $[0,1]$ is connected $\gamma([0,1]) \subseteq X$ is connected.
Hence either $\gamma([0,1]) \subseteq U$ or $\gamma([0,1]) \subseteq V$
This contradicts $\gamma(0) \in U$, $\gamma(1) \in V$. $\Box$

<u>Lemma 15.5</u> For $Y \subseteq \mathbb{R}$, Y is connected $\Leftrightarrow$ $\forall\, y_1, y_2 \in Y$ $[y_1, y_2] \subseteq Y$.

<u>Proof</u> ($\Leftarrow$) Suppose $y_1, y_2 \in Y$ and $[y_1, y_2] \subseteq Y$
Then $\gamma(t) = t y_1 + (1-t) y_2$ is a path in $[y_1, y_2] \subseteq Y$ from y_1 to y_2
$\Rightarrow$ Y is path connected hence connected.
($\Rightarrow$) Suppose Y is connected and $\exists\, y_1, y_2 \in Y$ with $[y_1, y_2] \not\subseteq Y$. Then $\exists\, z \in [y_1, y_2]$ st $z \notin Y$.
$\Rightarrow$ $Y \subseteq (-\infty, z) \cup (z, +\infty)$ with $y_1 \in (-\infty, z) \cap Y$
$y_2 \in (z, +\infty) \cap Y$. This contradicts connectedness of Y.

<u>Corollary 15.6</u> (Intermediate value theorem). Suppose X is connected $f : X \to \mathbb{R}$ continuous. If $y_1, y_2 \in f(X)$ (with $y_1 < y_2$)
Then $[y_1, y_2] \subseteq f(X)$.
<u>Proof</u> $f(X)$ is connected by 15.2. By 15.5 $[y_1, y_2] \subseteq f(X)$.

<u>Ex</u> $f : [0,1] \to [0,1]$, continuous. Then $\exists\, x \in [0,1]$ st $f(x) = x$.
<u>Reason</u> $g(x) = f(x) - x$ is continuous. $g(0) = f(0) - 0 \in [0,1]$
$\Rightarrow g(0) \geq 0$. Since $f(1) \in [0,1]$, $g(1) = f(1) - 1 \leq 0$.
$\Rightarrow g([0,1])$ contains 0 by 15.6. $\Box$