Recall: A topological space X is connected if $\forall\, U, V \subseteq X$ open with $U \cup V = X$, $U \cap V = \emptyset$, either $U = \emptyset$ or $V = \emptyset$.

A subset Y of a topological space X is connected if it's connected in the subspace topology. This amounts to: $\forall\, U, V \subseteq X$, open, with $Y \subseteq U \cap V$ and $(Y \cap U) \cap (Y \cap V) = \emptyset$ either $Y \cap U = \emptyset$ or $Y \cap V = \emptyset$

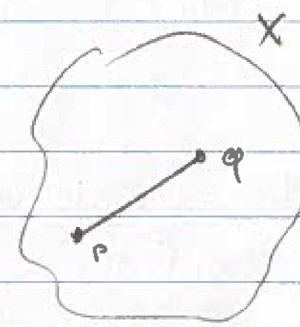
We proved:
- $[0,1]$ is connected
- if $f : X \to Y$ is continuous, X connected, then $f(X)$ is connected.

Def: A space X is path connected if $\forall\, p, q \in X$ $\exists\, \gamma : [0,1] \to X$, continuous with $\gamma(0) = p$, $\gamma(1) = q$.

We say: "γ is a path from p to q"

Ex: $\mathbb{R}^n$ is path connected: $\forall\, p, q \in \mathbb{R}^n$, $\gamma(t) = tp + (1-t)q$ is a continuous path from p to q.

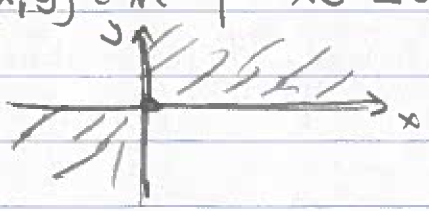
Ex: A subset $X \subseteq \mathbb{R}^n$ is convex if $\forall\, p, q \in X$ $tp + (1-t)q \in X$ $\forall\, t \in [0,1]$



Convex sets are path connected.

In particular closed and open balls in $\mathbb{R}^n$ (w.r.t. d_2, d_1, d_∞) are all convex, hence path connected

Ex: $X = \{(x,y) \in \mathbb{R}^2 \mid xy \geq 0\}$ is path connected, not convex

$X =$ 

Thm 15.4: path connected $\Rightarrow$ connected.

Proof: Suppose X is path connected, $U, V \subseteq X$ open $U \cup V = X$

$U \cap V = \emptyset$. Suppose both $U \neq \emptyset$ and $V \neq \emptyset$. Then
$\exists \gamma : [0,1] \to X$ s.t. $\gamma(0) \in U$, $\gamma(1) \in V$.
But then $[0,1] = \gamma^{-1}(U \cup V) = \gamma^{-1}(U) \cup \gamma^{-1}(V)$,
$0 \in \gamma^{-1}(U)$, $1 \in \gamma^{-1}(V)$ and $\gamma^{-1}(U) \cap \gamma^{-1}(V) = \emptyset$
This contradicts connectedness of $[0,1]$. $\square$

<u>Lemma 15.5</u> For $Y \subseteq \mathbb{R}$, Y is connected $\iff$ Y is convex: $\forall y_1, y_2 \in Y$, $y_1 \neq y_2$, $[y_1, y_2] \subset Y$. (Hence <u>for $Y \subseteq \mathbb{R}$</u> connected $\iff$ path connected)

<u>Proof</u> ($\Leftarrow$) If $\forall y_1, y_2 \in Y$ $[y_1, y_2] \subseteq Y$ then Y is path connected hence connected

($\Rightarrow$) Suppose Y is connected, and $\exists\, y_1, y_2 \in Y$ with $[y_1, y_2] \not\subseteq Y$.
Then $\exists\, z \in [y_1, y_2]$ st $z \notin Y$.
$\Rightarrow Y \subseteq (-\infty, z) \cup (z, +\infty)$ and $y_1 \in Y \cap (-\infty, z)$, $y_2 \in Y \cap (z, +\infty)$.
This contradicts connectedness of Y. $\square$

(IVT)

<u>Intermediate value Theorem</u> Suppose X is connected, $f : X \to \mathbb{R}$ continuous. Then $\forall y_1, y_2 \in f(X)$ with $y_1 < y_2$, $[y_1, y_2] \subseteq f(X)$.
In particular, $\forall y \in \mathbb{R}$ with $y_1 < y < y_2$ $\exists x \in X$ st $f(x) = y$.

<u>Proof</u> Since f is continuous and X is connected, $f(X) \subseteq \mathbb{R}$ is connected. Now apply 15.5.

<u>Ex</u> Suppose $f : [0,1] \to [0,1]$ is continuous. Then f has a <u>fixed point</u>: $\exists x \in [0,1]$ st $f(x) = x$.

<u>Proof</u> Consider $g(x) = f(x) - x$.

$g(0) = f(0) - 0 = f(0) \in [0,1]$. $\Rightarrow g(0) \geq 0$

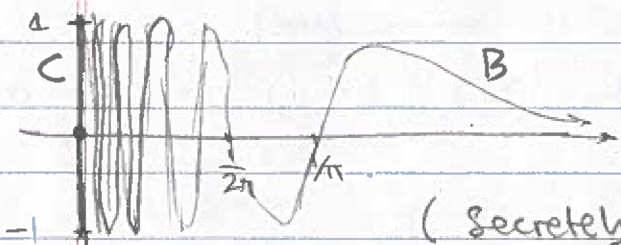
$g(1) = f(1) - 1 \leq 0$ since $f(1) \in [0,1]$.

Since g is continuous, $[0,1]$ is connected and $g(1) \leq 0 \leq g(0)$

$\exists x \in [0,1]$ s.t. $0 = g(0) = f(x) - x$. $\blacksquare$

Example of $A \subset \mathbb{R}^2$ which is connected and not path connected.

Let $B = \{(x, \sin\frac{1}{x}) \mid x > 0\}$. Since $f : [0,\infty) \to B$, $f(x) = (x, \sin\frac{1}{x})$ is continuous and surjective, B is path connected.



Let $A = (\{0\} \times [-1,1]) \cup B$

(Secretely $A = \overline{B}$, The closure of B).

Claim 1 A is connected.

Proof Suppose not: $\exists U_1, U_2 \subseteq \mathbb{R}^2$, open, st $A \subseteq U_1 \cup U_2$, $(A \cap U_1) \cap (A \cap U_2) = \emptyset$. and $A \cap U_1$, $A \cap U_2 \neq \emptyset$.

Since $C = \{0\} \times [-1,1]$, B are connected, $C \subseteq U_1$ and then $B \subset U_2$ (or the other way around). Say $C \subseteq U_1$.

Since $U_1 \subseteq \mathbb{R}^2$ is open $\exists r > 0$ st $B_r((0,0)) \subseteq U_1$

But $B_r((0,0)) \cap B \neq \emptyset \ \forall r \Rightarrow B \cap U_1 \neq \emptyset$.

Contradiction.

Claim 2 A is not path connected.

Suppose A is path connected. Then $\exists \gamma : [0,1] \to A$ st. $\gamma(0) = (0,0) \in C$, $\gamma(1) \in B$.

Since $\gamma(0) \in C$, $\gamma^{-1}(C) \neq \emptyset$

Since $B = \{(x,y) \in \mathbb{R}^2 \mid x > 0\} \cap A$, B is open in A.

$\Rightarrow \gamma^{-1}(B)$ is open in $[0,1]$

Since $\gamma^{-1}(C) = [0,1] \setminus \gamma^{-1}(B)$, $\gamma^{-1}(C)$ is closed in $[0,1]$

$\Rightarrow d = \sup(\gamma^{-1}(C))$ exists and is in $\gamma^{-1}(C)$, ie $\gamma(d) \in C$.

Now consider $\gamma|_{[d,1]} : [d,1] \to A$, $\gamma(t) = (x(t), y(t))$

Since γ is continuous so are $x(t)$, $y(t)$

For $t > d$, $\gamma(t) \in B$. $\Rightarrow \begin{cases} y(t) = \sin\left(\frac{1}{x(t)}\right). \\ x(t) > 0 \end{cases}$

We now argue: $\exists$ sequence $\{t_n\} \subset [d,1]$ s.t. $t_n \to d$ and $y(t_n) = (-1)^n$.

This would give us a contradiction. Continuity of γ implies $\gamma(t_n) = (x(t_n), y(t_n)) \longrightarrow (x(d), y(d)) = (0, y(d))$

On the other hand $y(t_n) = (-1)^n \not\longrightarrow y(d)$.

To construct $\{t_n\}$ first choose $\{s_n\} \subset (d,1)$ s.t. $s_n \to d$.

Since $s_n > d$ $\forall n$, $x(s_n) > 0$.

We can then always find $k \in \mathbb{N}$ s.t.

$$\frac{1}{\pi/2 + 2\pi k}, \quad \frac{1}{-\pi/2 + 2\pi k} < x(s_n).$$

In particular $\forall n$ $\exists u_n \in [0, x(s_n)]$ s.t.

$$\sin\left(\frac{1}{u_n}\right) = (-1)^n.$$

IVT $\Rightarrow$ $\exists$ $t_n \in [d, s_n]$ s.t. $x(t_n) = u_n$.

Then since $d \le t_n \le s_n$ and $s_n \to d$, $t_n \to d$

and $y(t_n) = (-1)^n$.

Theorem 16.1 If $U \subseteq \mathbb{R}^n$ is connected and open then U is path connected.

Proof. Define $\sim$ on U by $x \sim y \Leftrightarrow x$ and y can be connected by a path. Check that $\sim$ is an equiv relation.

Equiv classes of $\sim$ are disjoint.

Claim Any equiv class V of $\sim$ is open.

Proof Given $x \in V$ $\exists r > 0$ s.t. $B_r(x) \subseteq U$

But any point of $y \in B_r(x)$ can be connected to x by a path.

$\Rightarrow B_r(x) \subseteq V$. $\Rightarrow V$ is open.

Since U is connected there can be only one equiv class.

$\therefore$ U is path connected.