MATH 424 Eugene Lerman 02/28/2024

Differentiation

Definition $U \subseteq \mathbb{R}$ open, $f: U \to \mathbb{R}$. f is differentiable at $a \in U$ (f has a derivative at a) if $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists.

If f is differentiable at a, we write $f'(a)$ or $\frac{df}{dx}(a)$ for $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$.

Examples $U = \mathbb{R}$, $f(x) = x$. Then $\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} \frac{x-a}{x-a} = \lim 1 = 1$.

Thus $f(x) = x$ is differentiable at any $a \in \mathbb{R}$ and $(x)' = 1$.

$U = \mathbb{R}$, $f(x) = c$, a constant. Then f is differentiable at any $a \in \mathbb{R}$ since $\lim_{x\to a} \frac{c-c}{x-a} = \lim_{x\to a} 0 = 0$.

Ex $U = \{x \in \mathbb{R} \mid x \neq 0\}$, $f(x) = \frac{1}{x}$. For $a \in U$

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} \frac{1}{x-a}\left(\frac{1}{x} - \frac{1}{a}\right) = \lim_{x\to a} \frac{a-x}{(x-a)\,xa}$$

$$= \lim_{x\to a} \frac{-1}{xa} = -\frac{1}{a^2}$$

$$\therefore \left(\frac{1}{x}\right)' = -\frac{1}{x^2}.$$

Exercise (*) $\forall u, v \in \mathbb{R}$ $\forall n \in \mathbb{N}$ $u^n - v^n = (u-v)(u^{n-1} + u^{n-2}v + \dots + uv^{n-2} + v^{n-1})$

Ex $U = (0, \infty)$ $f(x) = x^{1/n}$, $n \in \mathbb{N}$.

$$f'(a) = \lim_{x\to a} \frac{1}{x-a}(x^{1/n} - a^{1/n}) \overset{(*)}{=} \lim_{x\to a} \frac{x^{1/n} - a^{1/n}}{(x^{1/n} - a^{1/n})(x^{\frac{n-1}{n}} + x^{\frac{n-2}{n}} a^{1/n} + \dots)}$$

$$= \lim_{x\to a} \left(\sum_{k=0}^{n-1} x^{\frac{n-1-k}{n}} a^{k/n}\right)^{-1} = \left(\sum_{k=0}^{n-1} a^{\frac{n-1-k}{n}} a^{\frac{k}{n}}\right)^{-1} = \left(n\, a^{\frac{n-1}{n}}\right)^{-1}$$

$$= \frac{1}{n} a^{\frac{1-n}{n}} = \frac{1}{n} a^{(1/n - 1)}$$

$$\therefore (x^{1/n})' = \frac{1}{n} x^{1/n - 1}$$

Equivalent definitions of differentiability of f at a:

1) $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ exists

2) $\exists f'(a)\in\mathbb{R}$ so that $\lim_{x\to a} \frac{f(x)-f(a)-f'(a)(x-a)}{x-a} = 0.$

Lemma 17.1 if $f: U\to\mathbb{R}$ is differentiable at a, then f is continuous at a.

Proof Suppose $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ exists. Call it $f'(a)$, as usual.

Then

$$\lim_{x\to a} f(x) = \lim_{x\to a}\left(f(a)+\frac{f(x)-f(a)}{x-a}(x-a)\right)$$

$$= f(a) + \lim_{x\to a}\frac{f(x)-f(a)}{x-a}\cdot\lim_{x\to a}(x-a) = f(a)+0\cdot f'(a) = f(a).$$

Lemma 17.2 Suppose f is differentiable at a, g is differentiable at $f(a)$. Then $g\circ f$ is differentiable at a and

$$(g\circ f)'(a) = g'(f(a))\cdot f'(a).$$

Proof

We need to show: $\lim_{x\to a}\frac{g(f(x))-g(f(a))}{x-a}$ exists and equals

$$g'(f(a))\cdot f'(a).$$

Let

$$h(y) = \begin{cases} \frac{g(y)-g(f(a))}{y-f(a)} & y\neq f(a) \\ g'(f(a)) & y = f(a)\end{cases}$$

Then h is continuous at $f(a)$ since $\lim_{y\to f(a)} h(y) = g'(f(a))$.

Moreover

$$\lim_{x\to a}\frac{g(f(x))-g(f(a))}{x-a} = \lim_{x\to a} h(f(x))\cdot\frac{f(x)-f(a)}{x-a} = h(f(a))\cdot f'(a)$$

$$= g'(f(a))\cdot f'(a).$$

$\square$

Corollary 17.3 Suppose $f(a) \neq 0$ and f is differentiable at a.
Then $k(x) = \frac{1}{f(x)}$ is differentiable at a and $k'(a) = -\frac{1}{(f(a))^2} f'(a)$

Proof $k(x) = (g \circ f)(x)$ where $g(y) = \frac{1}{y}$.

Theorem 17.4 Suppose f, g are differentiable at a, $c \in \mathbb{R}$ a constant.
Then cf, $f+g$, $f \cdot g$ are differentiable at a.
If $g(a) \neq 0$, f/g is also differentiable at a. Moreover

(i) $(cf)'(a) = c \cdot f'(a)$

(ii) $(f+g)'(a) = f'(a) + g'(a)$

(iii) $(f \cdot g)'(a) = f'(a) \cdot g(a) + f(a) \cdot g'(a)$

(iv) $\left(\frac{f}{g}\right)'(a) = \frac{f'(a)\, g(a) - f(a)\, g'(a)}{(g(a))^2}$ if $g(a) \neq 0$.

Proof Since $(c)' = 0$, (iii) $\Rightarrow$ (i).
(iii) + 17.3 $\Rightarrow$ (iv) since $\frac{f}{g} = f \cdot \frac{1}{g}$.

To prove (ii) observe that

$$\lim_{x \to a} \frac{(f+g)(x) - (f+g)(a)}{x-a} = \lim_{x \to a} \left(\frac{f(x)-f(a)}{x-a} + \frac{g(x)-g(a)}{x-a} \right) = f'(a) + g'(a)$$

For (iii) we compute

$$\lim_{x \to a} \frac{(f \cdot g)(x) - (f \cdot g)(a)}{x-a} = \lim_{x \to a} \frac{f(x)g(x) - f(x)g(a) + f(x)g(a) - f(a)g(a)}{x-a}$$

$$= \lim_{x \to a} f(x) \cdot \frac{g(x)-g(a)}{x-a} + \lim_{x \to a} \frac{f(x)-f(a)}{x-a} \cdot g(a)$$

$$= \underbrace{\lim_{x \to a} f(x)}_{\substack{\parallel \\ f(a) \text{ since } f \text{ is continuous at } a}} \cdot \lim_{x \to a} \frac{g(x)-g(a)}{x-a} + g(a) \lim_{x \to a} \frac{f(x)-f(a)}{x-a} = f(a)\, g'(a) + g(a)\, f'(a).$$

$\square$

Thm 17.5 Suppose $f: U \to \mathbb{R}$ is differentiable at a and a is a extremal point (a local max or a local min). Then $f'(a) = 0$.

Proof We prove the theorem when a is a local max.
Then $\exists\, \varepsilon > 0$ so that for $x \in (a-\varepsilon, x+a)$

$$f(x) - f(a) \le 0.$$

$\Rightarrow$ (i) for $x \in (a-\varepsilon, a)$ $\qquad \dfrac{f(x)-f(a)}{x-a} \ge 0$

(ii) for $x \in (a, a+\varepsilon)$ $\qquad \dfrac{f(x)-f(a)}{x-a} \le 0$

(i) $\Rightarrow \displaystyle\lim_{x \nearrow a} \frac{f(x)-f(a)}{x-a} \ge 0$, $\displaystyle\lim_{x \searrow a} \frac{f(x)-f(a)}{x-a} \le 0$

But $\displaystyle\lim_{x \to a} \frac{f(x)-f(a)}{x-a}$ exists and equals $f'(a)$. Hence

$$f'(a) \ge 0 \text{ and } f'(a) \le 0$$

$$\therefore\ f'(a) = 0$$

Ex $f(x) = x(1-x) = x - x^2$. Find $\sup\{f(x) \mid x \in [0,1]\}$ and $\inf\{f(x) \mid x \in [0,1]\}$

Solution Since $[0,1]$ is compact, f continuous, f attains sup and inf at some points $x_{max}, x_{min} \in [0,1]$.

If $x_{max}, x_{min} \in (0,1)$ then $f'(x_{max}) = f'(x_{min}) = 0$ by 17.5.

$f'(x) = 1 - 2x$. So $f'(x) = 0 \Leftrightarrow x = 1/2$, and $f(1/2) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

$f(0) = 0$, $f(1) = 0$.

$\Rightarrow \underset{f(0),\ f(1)}{\inf(f([0,1])) = 0}$, $\sup(f([0,1])) = 1/4$, $= f(1/2)$ $\qquad$ D
