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Last time
- Definition of $f: U \to \mathbb{R}$ being differentiable
- properties of derivatives, chain rule.

Note: $(x)' = 1$ and $(fg)' = f'g + fg' \Rightarrow (x^n)' = nx^{n-1}$ (by induction on n). $(x^{1/n})' = \frac{1}{n} x^{1/n - 1}$

$\rightsquigarrow$ get derivatives of rational powers of x.

- If f is differentiable at x_0 and x_0 an extremal point, then $f'(x_0) = 0$.

Rolle's Theorem $f: [a,b] \to \mathbb{R}$ continuous, differentiable on (a,b), $f(a) = f(b)$. Then $\exists\, c \in (a,b)$ so that $f'(c) = 0$.

Proof Since $[a,b]$ is compact, f achieves max and min on $[a,b]$. If a point $c \in (a,b)$ is extremal, we are done. Otherwise the extremal points are in $\{a,b\}$ and then f is constant hence $f'(c) = 0 \ \forall\, c \in (a,b)$. $\square$

Mean value Theorem Suppose $f: [a,b] \to \mathbb{R}$ is continuous, differentiable on (a,b). Then $\exists\, c \in (a,b)$ s.t.

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Note: This is Rolle's thm if $f(a) = f(b)$.

Proof of MVT

Let $h(x) = \frac{f(b) - f(a)}{b-a} \cdot (x - a)$. Note: $h'(x) = \frac{f(b) - f(a)}{b - a} \ \forall x$

Let

$$g(x) = f(x) - h(x).$$

Then
$$g(a) = f(a) - h(a) = f(a)$$
$$g(b) = f(b) - \frac{f(b) - f(a)}{\cancel{b-a}} \cdot \cancel{(b-a)} = f(a)$$

By Rolle's thm applied to $g(x)$ we get: $\exists\, c \in (a,b)$ so that $0 = g'(c)$.

But $g'(c) = f'(c) - h'(c) = f'(c) - \dfrac{f(b) - f(a)}{b - a}$.

Thus $\exists c \in (a,b)$ s.t. $f(c) = \frac{f(b)-f(a)}{b-a}$.

Corollary 18.1 Suppose $f : (a,b) \to \mathbb{R}$ is differentiable and $f'(x) = 0 \ \forall x \in (a,b)$. Then f is constant.

Proof If f is not constant, $\exists x_1, x_2 \in (a,b)$, $x_1 < x_2$ s.t. $f(x_1) \neq f(x_2)$. By mean value Thm $\exists c \in (a,b)$ s.t.

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \neq 0.$$

Contradiction. □

Corollary 18.2 $f, g : (a,b) \to \mathbb{R}$ differentiable, $f'(x) = g'(x) \ \forall x$. Then $\exists c \in \mathbb{R}$ s.t. $f(x) = g(x) + c$.

Proof $(f-g)'(x) = f'(x) - g'(x) = 0$.

By 18.1 $h(x) = f(x) - g(x)$ is constant. □

Ex Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function. Suppose $\exists \alpha > 1$ st

$$|f(x) - f(y)| \leq |x-y|^{\alpha} \quad \text{for all } x, y \in \mathbb{R}.$$

Then f is constant.

Proof It's enough to prove that f is differentiable and that $f'(a) = 0 \ \forall a$. So fix $a \in \mathbb{R}$. $\forall x \neq a$

$$\left| \frac{f(x) - f(a)}{x-a} - 0 \right| = \frac{|f(x) - f(a)|}{|x-a|} \leq |x-a|^{\alpha - 1}$$

Given $\varepsilon > 0$ let $\delta = \varepsilon^{\frac{1}{\alpha-1}}$. Then, for $x \neq a$

$$|x-a| < \delta \Rightarrow \varepsilon = \delta^{\alpha-1} > |x-a|^{\alpha-1} \geq \left| \frac{f(x)-f(a)}{x-a} - 0 \right|$$

$$\therefore \lim_{x \to a} \frac{f(x) - f(a)}{x-a} = 0.$$ □

<u>Lemma 18.3</u> Suppose $f:(a,b)\to\mathbb{R}$ is differentiable and $f'(x)$ is bounded. Then f is uniformly continuous.

<u>Proof</u> Let M be an upper bound for $\{|f'(x)| \mid x\in(a,b)\}$.
By the Mean value theorem $\forall x, y \in (a,b)$, $x<y$
$\exists c \in (x,y)$ st $\dfrac{f(y)-f(x)}{y-x} = f'(c)$

$$\Rightarrow \frac{|f(y)-f(x)|}{|y-x|} = |f'(c)| \le M$$

$$\Rightarrow |f(y)-f(x)| \le M\,|y-x| \qquad (*)$$

Therefore given $\varepsilon>0$ let $\delta = \varepsilon/M$. Then

$$|y-x|<\varepsilon/M \Rightarrow |f(y)-f(x)| \le M\,|y-x| < M\frac{\varepsilon}{M} = \varepsilon. \quad \square$$

<u>Remark</u> $f(x)=\sqrt{x}$ is continuous on $[0,1]$ hence uniformly continuous on $[0,1)$ ($[0,1]$ is compact). Hence it's uniformly continuous on $(0,1)$. But $f'(x) = \dfrac{1}{2\sqrt{x}}$ is <u>not</u> bounded on $(0,1)$.

<u>Thm 18.4</u> Suppose $f:(a,b)\to\mathbb{R}$ is differentiable on (a,b). Then

(i) $f'(x)>0 \ \forall x \Rightarrow f$ is strictly increasing

(ii) $f'(x)<0 \ \forall x \Rightarrow f$ is strictly decreasing

(iii) $f'(x)\ge 0 \ \forall x \Leftrightarrow f$ is nondecreasing

(iv) $f'(x)\le 0 \ \forall x \Leftrightarrow f$ is non increasing

<u>Proof of</u> (iii). $(\Rightarrow)$ Suppose $f'(z)\ge 0 \ \forall z\in(a,b)$, $x,y\in(a,b)$ $x<y$. Then $\exists c\in(x,y)$ st

$$\frac{f(y)-f(x)}{y-x} = f'(c) \overset{\text{assumption}}{\ge} 0$$

$$\Rightarrow f(y)-f(x) = f'(c)\,(y-x) \ge 0$$

$(\Leftarrow)$ If f is nondecreasing then $\forall x<y$

$$\frac{f(y)-f(x)}{y-x} \ge 0. \Rightarrow f'(y)\ge 0 \ \forall y$$

Remark $f(x) = x^3$ is strictly increasing but $f'(0)=0$.
So strictly increasing $\not\Rightarrow$ f' is positive.

Inverse function theorem Suppose $f:(a,b)\to(c,d)$ is a continuous bijection, $x_0\in(a,b)$, f is differentiable at x_0 and $f'(x_0)\neq 0$. Then $f^{-1}:(c,d)\to(a,b)$ is differentiable at $y_0=f(x_0)$ and

$$(*)\quad (f^{-1})'(y_0)\cdot f'(x_0) = 1 \qquad \left(\text{ie. } (f^{-1})'(y_0) = \frac{1}{f'(x_0)}\right)$$

Remark The content of the theorem is that f^{-1} is differentiable at y_0. The formular $(*)$ then follows from
$$f^{-1}(f(x)) = x :$$
chain rule $\Rightarrow$
$$1 = (f^{-1}\circ f)'(x_0) = (f^{-1})'(f(x_0))\cdot f'(x_0).$$

To prove the inverse function thm we'll need to know that f^{-1} is continuous at all points of (c,d).

Lemma (S,d), (S',d') metric spaces, S compact, $f: S\to S'$ continuous bijection. Then $f^{-1}: S'\to S$ is continuous.

Proof $g := f^{-1}$ is continuous $\Leftrightarrow$ $\forall$ closed $C\subseteq S$, $g^{-1}(C)$ is closed.
Now, since $g=f^{-1}$, $g^{-1}(C) = f(C)$.
Since $C\subseteq S$ is closed, C is compact. f continuous $\Rightarrow f(C)$ is compact. But S' is a metric space, so $f(C)$ is closed
$\therefore$ $g=f^{-1}$ is continuous.