MATH 424 Eugene Lerman

03/04/24
19.1

Last time: If (S,d), (S',d') are metric spaces, S compact, $f: S \to S'$ continuous bijection then $f^{-1}: S' \to S$ is continuous.

Inverse function Theorem $f: (a,b) \to (c,d)$ continuous bijection, $x_0 \in (a,b)$, f differentiable at x_0 and $f'(x_0) \neq 0$. Then $f^{-1}: (c,d) \to (a,b)$ is differentiable at $y_0 = f(x_0)$ and

$$(f^{-1})'(y_0) \cdot f'(x_0) = 1, \text{ i.e. } (f^{-1})'(f(x_0)) = \frac{1}{f'(x_0)}.$$

Proof We first argue that $f^{-1}: (c,d) \to (a,b)$ is continuous:

Given $y_0 \in (c,d)$, $x_0 = f^{-1}(y_0) \in (a,b)$. $\Rightarrow$ $\exists\, \varepsilon > 0$ so that $[x_0-\varepsilon, x_0+\varepsilon] \subseteq (a,b)$.

Since $[x_0-\varepsilon, x_0+\varepsilon]$ is compact, $f: [x_0-\varepsilon, x_0+\varepsilon] \to f([x_0+\varepsilon, x_0-\varepsilon])$ has a continuous inverse. In particular f^{-1} is continuous at $y_0 = f(x_0) \in f([x_0-\varepsilon, x_0+\varepsilon])$.

We now argue: $\lim_{y \to y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0}$ exists and equals $\frac{1}{f'(x_0)}$

Enough to show:

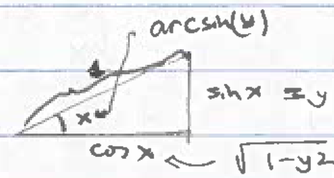
$\forall$ sequence $\{y_n\} \subseteq (c,d) \setminus \{y_0\}$ with $y_n \to y_0$ $\lim_{n\to\infty} \frac{f^{-1}(y_n) - f^{-1}(y_0)}{y_n - y_0}$ exists and equals $\frac{1}{f'(x_0)}$.

Let $x_n = f^{-1}(y_n)$. Since f^{-1} is continuous, $x_n = f^{-1}(y_n) \to f^{-1}(y_0) = x_0$

Therefore

$$\lim_{n\to\infty} \frac{f^{-1}(y_n) - f^{-1}(y_0)}{y_n - y_0} = \lim_{n\to\infty} \frac{x_n - x_0}{f(x_n) - f(x_0)} = \frac{1}{f'(x_0)}.$$

We are done. $\square$



Ex $f(x) = \sin x \quad x \in (-\pi/2, \pi/2)$

$g(y) = f^{-1}(y) = \arcsin(y)$

$$g'(\sin x) = \frac{1}{f'(x)} = \frac{1}{\cos x} = \frac{1}{\cos(\arcsin y)} = \frac{1}{\sqrt{1-y^2}}$$

Definition $f:(a,b)\to\mathbb{R}$ is twice differentiable if f is differentiable and f' is differentiable. We write $f''(x)$ for $(f'(x))'$

Similarly f is k-times differentiable ($k\geq 1$) if f is $(k-1)$-times differentiable (so $f^{(k-1)}$ exists) and $f^{(k-1)}$ is differentiable.

f is infinitely differentiable if f is k-times differentiable for all $k\geq 1$.

Notation $C^k(a,b) = \{ f:(a,b)\to\mathbb{R} \mid f$ is k-times differentiable and $f^{(k)}$ is continuous $\}$

$$C^\infty(a,b) := \{ f:(a,b)\to\mathbb{R} \mid f\in C^k(a,b) \text{ for all } k\} = \bigcap_{k=1}^{\infty} C^k(a,b)$$

Theorem (Taylor's theorem, finite Taylor series)

Let $U\subseteq\mathbb{R}$ be an open interval, $f:U\to\mathbb{R}$ n-times differentiable. Fix $a\in U$. Then $\forall x\in U$ $\exists$ $c\in$ between a and x so that

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k + \overbrace{\frac{f^{(n)}(c)}{n!}(x-a)^n}^{\text{error term}}.$$

Note: if $n=1$, this says: $f(x) = \overbrace{\frac{f^{(0)}(a)}{\underbrace{0!}_{1=}}}^{=f(a)}\underbrace{(x-a)^0}_{=1} + \frac{f'(c)}{1!}(x-a)^1$

for c between x & a, ie

$$f'(c) = \frac{f(x)-f(a)}{x-a}, \quad \text{which is MVT.}$$

Note also $f^{(n)}(a)=0$ $\forall n$ $\not\Rightarrow$ f is constant

Ex

$$f(x) = \begin{cases} e^{-1/x} & x>0 \\ 0 & x\leq 0 \end{cases}$$

Since $\lim\limits_{x\searrow 0} e^{-1/x} = \lim\limits_{x\searrow 0} \frac{1}{e^{1/x}} = 0$, f is continuous at 0.

$$f'(0) = \lim_{x\searrow 0}\frac{e^{-1/x}-0}{x} = \lim_{x\searrow 0}\frac{1}{x\,e^{1/x}} = \lim_{y\to+\infty}\frac{y}{e^y} \overset{\text{l'Hopital}}{=} \lim_{y\to+\infty}\frac{1}{e^y} = 0$$

$\therefore f$ is differentiable at 0 and $f'(0) = 0$.

Induction on $n \Rightarrow f$ is n-times differentiable at 0 and $f^{(n)}(0) = 0$

(away from zero f is infinitely differentiable)

$\therefore f$ is C^∞ and $f^{(n)}(0) = 0 \quad \forall n$.

Taylor's thm says: $\forall x > 0 \ \exists \ c \in (0, x)$ s.t.

$$e^{-1/x} = 0 + \frac{f^{(n)}(c)}{n!} x^n$$

, which is not all that useful.

On the other hand Taylor's theorem has the following useful corollary:

__Corollary 19.1__ Suppose $f \in C^\infty(-a, a)$ and $\exists \ M, C > 0$ so that $\forall k \in \mathbb{N} \quad \forall x \in (-a, a) \qquad |f^{(k)}(x)| \leq M C^k$

Then $\forall x \in (-a, a)$

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \quad \left(:= \lim_{N \to \infty} \sum_{k=0}^{N} \frac{f^{(k)}(0)}{k!} x^k \right)$$

__Example__ $f(x) = \sin x$, $f'(x) = \cos x$ $\quad f''(x) = -\sin x$

$f^{(3)}(x) = -\cos x \qquad f^{(4)}(x) = \sin x \quad \ldots$

All derivatives satisfy $|f^{(n)}(x)| \leq 1 \quad \forall x$ (so $M = C = 1$)

Corollary 19.1 $\Rightarrow$

$$\sin(x) = \sum_{k=0}^{\infty} \frac{\sin^{(k)}(0)}{k!} x^k = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}.$$

__Proof of Taylor's theorem__

It's no loss of generality to assume: $0 \in U$, $a = 0$.

Fix $n \geq 1$, fix $x \in U \setminus \{0\}$.

Let
$$E = \left(f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} x^k \right) \frac{n!}{x^n}$$

We want to show: $\exists \ c$ between 0 and x s.t.

$$f^{(n)}(c) = E.$$

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Let $g(t) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} t^k - \frac{E \cdot t^n}{n!}$

Then

$$g(0) = f(0) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} 0^k - \frac{E \cdot 0^n}{n!} = f(0) - f(0) = 0$$

$$g(x) = \left(f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} x^k\right) - E \cdot \frac{x^n}{n!} = \left(f(x) - \sum \frac{f^{(k)}(0)}{k!} x^k\right) - \left(f(x) - \sum \frac{f^{(k)}(0)}{k!} x^k\right) =$$

$$g'(0) = f'(0) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} k t^{k-1} \Big|_{t=0} - \frac{E}{n!} n t^{n-1} \Big|_{t=0} = 0$$

Similarly $g''(0) = g^{(3)}(0) = \ldots = g^{(n-1)}(0) = 0$

Note $g^{(n)}(t) = f^{(n)}(t) - E$.

Thus if $g^{(n)}(c) = 0$ for some c between 0 & x, then $E = f^{(n)}(c)$ between 0 & x and we're done.

Consider the case $x > 0$ (the case $x < 0$ is similar)

Since $g(0) = 0 = g(x)$, Rolle's thm $\Rightarrow \exists\, c_1 \in (0, x)$ s.t. $g'(c_1) = 0$

Since $g'(0) = 0$, $g'(c_1) = 0$, Rolle's thm $\Rightarrow \exists\, c_2 \in (0, c_1)$ s.t. $g''(c_2) = 0$

$\vdots$

$\exists\, c \in (0, c_{n-1})$ s.t. $g^{(n)}(c) = 0$ and we're done. $\square$