MATH 424 Eugene Lerman 01/19/2024

2.1

Recall For $\emptyset \neq S \subseteq \mathbb{R}$, an upper bound of S is $a \in \mathbb{R}$ so that $s \leq a \ \forall s \in S$ (it may or may not exist)

A least upper bound (supremum) is $\sup S \in \mathbb{R}$ so that

1) $\sup S$ is an upper bound of S and

2) $\forall$ upper bound a of S, $\sup S \leq a$.

"Completeness axiom": any subset of $\mathbb{R}$ bounded above has a least upper bound.

Notation We may write $\sup S = \infty$ if S is not bounded above.

Similarly one can define:

lower bound, infimum $\equiv$ greatest lower bound.

Not hard to show: $\inf(S) = -\sup(-S)$ where $-S = \{-s \mid s \in S\}$

Hence completeness axiom $\Rightarrow \forall S \subseteq \mathbb{R}, S \neq \emptyset$ so that S is bounded below, $\inf(S)$ exists.

Lemma 2.1 $\forall x \in \mathbb{R} \ \exists n \in \mathbb{N}$ so that $x < n$.

Proof Suppose not: $\exists x_0 \in \mathbb{R}$ so that $n \leq x_0 \ \forall n \in \mathbb{N}$.

Then $\mathbb{N}$ is bounded above by x_0. By completeness axiom $a = \sup \mathbb{N}$ exists. $\Rightarrow \forall n \in \mathbb{N}$, $n+1 \in \mathbb{N}$ hence $n+1 \leq a$

$\Rightarrow n \leq a-1 \ \forall n \in \mathbb{N}$.

$\Rightarrow a-1$ is an upper bound of $\mathbb{N}$. Contradiction since

$a-1 < a = \sup \mathbb{N}$. $\square$

Cor 2.2 $\forall \varepsilon > 0 \ \exists n \in \mathbb{N}$ s.t $\frac{1}{n} < \varepsilon$.

Proof By 2.1 $\exists n \in \mathbb{N}$ s.t $\frac{1}{\varepsilon} < n$. $\square$

Cor 2.3 $\forall x \in \mathbb{R} \ \exists n \in \mathbb{Z}$ s.t $n \leq x < n+1$

Pf See p 26

Theorem 2.4 ($\mathbb{Q}$ is dense in $\mathbb{R}$) $\forall x \in \mathbb{R} \ \forall \varepsilon > 0 \ \exists r \in \mathbb{Q}$ s.t $|x - r| < \varepsilon$.

Proof By 2.2 $\exists N \in \mathbb{N}$ s.t. $\frac{1}{N} < \varepsilon$. By 2.3 $\exists n \in \mathbb{N}$ st

$$n \leq Nx < n+1$$

$$\Rightarrow \frac{n}{N} \le x < \frac{n+1}{N} = \frac{n}{N} + \frac{1}{N} \Rightarrow \quad 0 \le x - \frac{n}{N} < \frac{1}{N} < \varepsilon$$

$$\therefore \quad \left| x - \frac{n}{N} \right| = x - \frac{n}{N} < \varepsilon. \qquad \square$$

Exercise 2.4 is equivalent to: $\forall a, b \in \mathbb{R}, a < b \ \exists r \in \mathbb{Q}$ s.t $a < r < b$.

Theorem 2.5 $\forall a > 0 \ \exists! x > 0$ s.t $x^2 = a$ (cf p28)

Proof (uniqueness). $\forall x_1, x_2 > 0$, $x_1 < x_2 \Rightarrow x_1^2 < x_1 x_2 < x_2^2$

$\Rightarrow \exists$ at most one $x > 0$ s.t $x^2 = a$.

(Existence) Consider $S = \{ b \in \mathbb{R} \mid 0 \le b, \ b^2 \le a \}$.

Since $a > 0$, $0 \in S \Rightarrow S \ne \emptyset$.

If $c > \max(a, 1)$, $c^2 > a \cdot 1 = a$. $\Rightarrow \forall b \in S$, $b < c$.

$\Rightarrow$ S is bounded above. Hence $\sup S$ exists. Let $y = \sup S$

We now argue: $y^2 = a$.

Since $0 < \min\{a, 1\}$ and since $(\min\{a,1\})^2 \le \min\{a,1\} \cdot 1 \le a$.

S contains a positive number, namely $\min\{a, 1\}$.

$\Rightarrow y = \sup S > 0$.

For any ε s.t $0 < \varepsilon < y$, $\boxed{(y-\varepsilon)^2 < y^2 < (y+\varepsilon)^2}$

Since $y = \sup S$, $y + \varepsilon \notin S$. Since $y - \varepsilon < y = \sup S$ $\exists b \in S$ s.t $y - \varepsilon < b \le y$

$\Rightarrow (y-\varepsilon)^2 < b^2 \le a$ ($b \in S$!)

Since $y + \varepsilon > 0$ and $y + \varepsilon \notin S$, $(y+\varepsilon)^2 > a$.

$\Rightarrow \boxed{(y-\varepsilon)^2 < a < (y+\varepsilon)^2}$

$$\Rightarrow \quad (y-\varepsilon)^2 - (y+\varepsilon)^2 < y^2 - a < (y+\varepsilon)^2 - (y-\varepsilon)^2$$

$$\Rightarrow \quad -4y\varepsilon < y^2 - a < 4y\varepsilon$$

$$\Rightarrow \quad |y^2 - a| < 4y\varepsilon. \quad \text{for any } \varepsilon \text{ with } 0 < \varepsilon < y.$$

Note: If $d \in \mathbb{R}$ $d \ge 0$ and $d < \delta$ for any $\delta > 0$, then $d = 0$.

This is because: if $d \ne 0$ $\exists N \in \mathbb{N}$ s.t $0 < \frac{1}{N} < d$.

Finally given $\delta > 0$ take $\varepsilon > 0$ with $\varepsilon < \min(y, \frac{\delta}{4y})$

Then $0 < \varepsilon < y$ and $|y^2 - a| < 4y\varepsilon \le 4y \cdot \frac{\delta}{4y} = \delta$

Since δ is arbitrary $|y^2 - a| = 0. \Rightarrow y^2 = a.$ $\square$

Remark This is a long and complicated proof of a fact that will become trivial once we know that $f(x) = x^2$ is continuous and once we prove the intermediate value theorem.

The reason the book proves this is: the positives in $\mathbb{R}$ are exactly the set $\{a^2 \mid a \in \mathbb{R}, a \neq 0\}$. Thus the function $\cdot : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ uniquely determines the order structure of $\mathbb{R}$

Decimal representations of the reals.

Definition Given $a_0 \in \mathbb{Z}, n \ge 0, a_1 \ldots a_n \in \{0, \ldots 9\}$ the book defines

$$a_0.a_1 \ldots a_n := a_0 + \frac{a_1}{10} + \cdots + \frac{a_n}{10^n}.$$

Note if $a_0 = -2$ $n = 1$, $a_1 = 1$ then $(-2).1 := -2 + \frac{1}{10}$

and not -2.1 $-(2 + \frac{1}{10})$

Next the book makes sense of infinite decimal expressions:

$a_0 \in \mathbb{Z}$, $a_1, a_2 \ldots a_n, \ldots$ infinite sequence of digits (i.e. elements of $\{0, 1, \ldots, 9\}$)

It would like to define

$$a_0.a_1 a_2 \ldots := a_0 + \sum_{i=1}^{\infty} \frac{a_i}{10^i}$$

But

$$\sum_{i=1}^{\infty} \frac{a_i}{10^i} = \lim_{N \to \infty} \sum_{i=1}^{N} \frac{a_i}{10^i}$$ and limits haven't been defined yet. Nor is there a proof that such a limit exists. So the work around is:

Consider the set $S = \{a_0, \underline{a_0.a_1}, a_0.a_1a_2, \ldots a_0.a_1 \ldots a_n, \ldots\}$

$$\equiv \{a_0.a_1 \ldots a_n \mid n \geq 0\}$$

Argue that S is bounded above. Then define

$a_0.a_1a_2 \ldots a_n \ldots = \sup S.$

Next the book argues:

$\forall\ a \in \mathbb{R} \quad \exists\ a_0 \in \mathbb{Z},\ a_1, a_2, \ldots a_n \ldots \qquad a_i \in \{0, 1, \ldots 9\}$

so that

$$a = a_0.a_1a_2 \ldots$$

<u>Note</u> such a representation is not unique since

$0.999\ldots \quad = \sup\{0.\underbrace{9\ldots9}_{n} \mid n \geq 1\} = 1 = 1.00\ldots$

This kind of ambiguity is the only kind of ambiguity:

if $a = a_0.a_1 \ldots a_n, \quad a_n \neq 0$

$a = a_0.a_1 \ldots (a_n - 1)\,99\ldots 9$

The book then uses the fact that any real number has an essentially unique decimal expression to argue that

any two complete ordered fields are "the same"

i.e if $(\mathbb{R}, +, \cdot)$ and $(\mathbb{R}', +', \cdot')$ are two complete ordered fields then $\exists$ a bijection

$$\varphi : \mathbb{R} \longrightarrow \mathbb{R}'$$

s.t $\varphi(0) = 0'$, $\varphi(1) = 1'$ $\quad \varphi(x+y) = \varphi(x) +' \varphi(y)$

and $\varphi(x \cdot y) = \varphi(x) \cdot' \varphi(y)$.

I am skeptical that the argument in the book is rigorous.