<u>Taylor's theorem</u> $U \subseteq \mathbb{R}$ open interval (i.e. connected open set)

$f: U \to \mathbb{R}$ n-times differentiable. Fix $a \in U$. For any $x \in U$

$\exists\ c = c(a,x)$ between x and a so that

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(n)}(c)}{n!}(x-a)^n.$$

<u>Remark</u> if $n=1$ this says: $\exists\ c$ between a and x s.t.

$$f(x) = \frac{f(a)}{\underset{1}{0!}}\underbrace{(x-a)^0}_{=1} + \frac{f'(c)}{1!}(x-a)^1$$

i.e. $\exists\ c$ s.t. $f'(c) = \frac{f(x)-f(a)}{x-a}$, which is MVT.

<u>Proof of Taylor's Theorem</u> No loss of generality to assume $0 \in U$ and $a=0$. Fix $n \geq 1$, $x \in U \setminus \{0\}$.

Let

$$E = \left(f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} x^k\right)\frac{n!}{x^n}.$$

We want to show: $\exists\ c$ between 0 and x st

$$f^{(n)}(c) = E$$

Let

$$g(t) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} t^k - E \cdot \frac{t^n}{n!}$$

Then

$$g(0) = f(0) - \left(\frac{f(0)}{0!} + \sum_{k=1}^{n-1} \frac{f^{(k)}(0)}{k!} 0^k - \frac{E}{n!} \cdot 0\right) = f(0) - f(0) = 0$$

$$g(x) = \left(f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} x^k\right) - E\frac{x^n}{n!} =$$

$$= \left(f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} x^k\right) - \left(f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} x^k\right) = 0$$

$$g'(0) = f'(0) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} k\, t^{k-1}\Big|_{t=0} - \frac{E}{n!} n \cdot t^{n-1}\Big|_{t=0}$$

$$= f'(0) - f'(0) = 0$$

Similarly, $g''(0) = g^{(3)}(0) = \cdots = g^{(n-1)}(0) = 0$,
while
$$g^{(n)}(t) = f^{(n)}(t) - E.$$
Thus <u>if</u> $g^{(n)}(c) = 0$ for some c between 0 and x, then
$$0 = f^{(n)}(c) - E, \text{ i.e. } E = f^{(n)}(c),$$
which is what we want to prove.

We consider the case $x > 0$ (the case $x < 0$ is similar).

Since $g(0) = 0 = g(x)$, Rolle's thm implies
$\exists\, c_1 \in (0, x)$ s.t. $g'(c_1) = 0$.

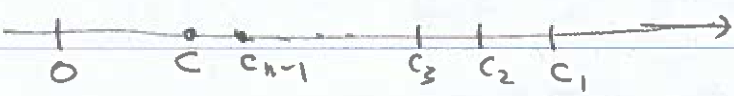
Since $g'(0) = 0$, $g'(c_1) = 0$, Rolle's thm $\Rightarrow \exists\, c_2 \in (0, c_1)$ st
$$g''(c_2) = 0$$

Since $g''(0) = 0$, $g''(c_2) = 0$, $\exists\, c_3 \in (0, c_2)$ st $g^{(3)}(c_3) = 0$

$\vdots$

$\exists\, c \in (0, c_{n-1})$ s.t. $g^{(n)}(c) = 0$.

And we're done



$\square$

<u>Corollary 19.1</u> Suppose $f \in C^\infty(-a, a)$ and $\exists\, M, C > 0$ s.t.
$$|f^{(k)}(x)| \le M \cdot C^k \quad \forall k \quad \forall x \in (-a, a)$$
Then $\forall x \in (-a, a)$
$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

In fact $S_N(x) := \sum_{k=0}^{N} \frac{f^{(k)}(0)}{k!} x^k$ converges uniformly to f on $(-a, a)$.

<u>Remark</u> We can replace $(-a, a)$ with $(x_0 - a, x_0 + a)$. Then
$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k.$$

Proof By Taylor's Theorem

$$f(x) - S_N(x) \equiv f(x) - \sum_{k=0}^{N} \frac{f^{(k)}(0)}{k!} x^k = \frac{f^{(N+1)}(c)}{(N+1)!} x^{N+1}$$

for some c between 0 and x.

By assumption

$$|f(x) - S_N(x)| = \frac{|f^{(N+1)}(c)|}{(N+1)!} |x|^{N+1} \leq \frac{M \cdot C^{N+1}}{(N+1)!} a^{N+1}$$

Claim $\lim_{N \to \infty} \frac{M C^{N+1}}{(N+1)!} a^{N+1} = 0$

Proof of claim Let $b_N = \frac{M C^N a^N}{N!}$. Note $b_N \geq 0$

Then $\frac{b_{N+1}}{b_N} = \frac{N!}{(N+1)!} \frac{M (Ca)^{N+1}}{M (Ca)^N} = \frac{Ca}{N+1} \xrightarrow[N \to \infty]{} 0$

$\Rightarrow \exists N_0$ st for $\forall n \geq N_0$ $\frac{b_{n+1}}{b_n} \leq \frac{1}{2}$, ie $b_{n+1} \leq \frac{1}{2} b_n$

$\Rightarrow \forall k$

$$b_{n+k} \leq \frac{1}{2} b_{n+(k-1)} \leq \dots \leq \frac{1}{2^k} b_n$$

$\Rightarrow \lim_{N \to \infty} b_N = 0$

Note: Claim implies $S_N(x) \to f(x)$ uniformly, so we're done. □

Ex $f(x) = \cos x$ Since $|f^{(k)}(x)| \leq 1 \quad \forall k$

Corollary applies. $\Rightarrow$

$$\cos x = \sum_{k=0}^{\infty} \frac{\cos^{(k)}(0)}{k!} x^k = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

for all x.

Definition A function f is real analytic on an open set $U \subseteq \mathbb{R}$ if f is infinitely differentiable on U and $\forall a \in U \quad \exists \delta > 0$ st

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k \quad \forall x \in (a-\delta, a+\delta)$$

$f(x) = \sin(x)$, $\cos x$, e^x, polynomials are real analytic on $\mathbb{R}$

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$ is not real analytic at 0

since $f^{(n)}(0) = 0 \ \forall n$ but $f(x) \not\equiv 0$ for x near 0.

A function $f \in C^\infty(\mathcal{U})$ with the property that $f^{(k)}(a) = 0 \ \forall k$ ($a \in \mathcal{U}$ fixed) is call flat at a.

Ex $f(x) = \frac{1}{1+x}$ is real analytic on $\mathbb{R} \setminus \{-1\}$.

Proof Recall that $(q-1)(1+q+\dots+q^n) = q^{n+1}-1$.

$$\Rightarrow \sum_{k=0}^{n} q^n = \frac{q^{n+1}-1}{q-1} \xrightarrow[n\to\infty]{} \frac{1}{1-q} \quad \text{if } |q|<1.$$

Thus for q with $|q|<1$ $\sum_{k=0}^{\infty} q^n = \frac{1}{1-q}$. $\Rightarrow \sum_{n=0}^{\infty} (-1)^n q^n = \frac{1}{1+q}$

Now, for $a \neq -1$

$$\frac{1}{1+x} = \frac{1}{1+a+x-a} = \frac{1}{1+a} \cdot \frac{1}{1+\frac{x-a}{a+1}} = \frac{1}{1+a} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n (x-a)^n}{(a+1)^n}$$

if $\left|\frac{x-a}{a+1}\right| < 1$

Thus $\forall a$, $\forall x$ with $|x-a| < |a+1|$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(1+a)^{n+1}} (x-a)^n$$

And one can show that $f^{(k)}(a) = \frac{(-1)^k}{(1+a)^{k+1}} k!$ ...