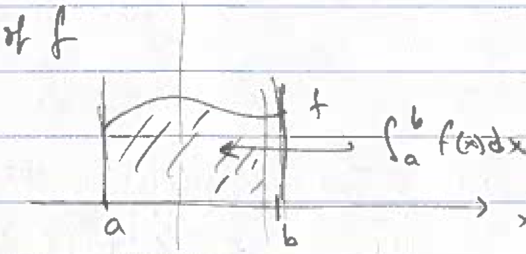
Integration

If $[a,b] \subseteq \mathbb{R}$ is an integral, $f:[a,b] \to \mathbb{R}$ non negative function Then

$$\int_a^b f(x)\,dx = \text{area under the graph of } f$$



There are several ways to define $\int_a^b f(x)\,dx$. We'll discuss to equivalent constructions: the Darboux integral and the Riemann integral.

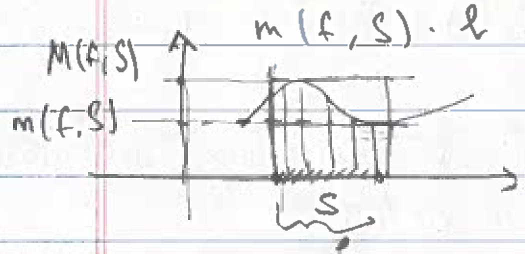
Notation $f:[a,b] \to \mathbb{R}$ bounded function, $S \subseteq [a,b]$ a subset $(S \neq \emptyset)$.

$$M(f,S) := \sup\{f(x) \mid x \in S\}$$

$$m(f,S) = \inf\{f(x) \mid x \in S\}.$$

Note If S is an interval of length ℓ, $f|_S \geq 0$ we expect

$$m(f,S)\cdot \ell \leq \int_S f(x)\,dx \leq M(f,S)\cdot \ell$$



Definition (i) A partition P of an interval $[a,b]$ is a finite strictly increasing sequence

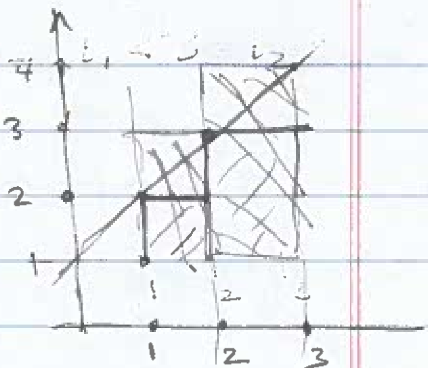
$$a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$$

(ii) The upper Darboux sum $U(f,P)$ of $f:[a,b] \to \mathbb{R}$ w.r.t P is

$$U(f,P) := \sum_{k=1}^{n} M(f,[t_{k-1},t_k])\cdot(t_k - t_{k-1})$$

(iii) The lower Darboux sum $L(f,P)$ of $f:[a,b] \to \mathbb{R}$ is

$$L(f,P) := \sum_{k=1}^{n} m(f,[t_{k-1},t_k])\cdot(t_k - t_{k-1})$$



Ex $\quad f(x) = x+1 \qquad [a,b] = [1,3]$

$P = 1 = t_0 < 2 = t_1 < 3 = t_2$

$U(f,P) = 3\cdot(2-1) + 4\cdot(3-2) = 3+4 = 7$

$L(f,P) = 2\cdot(2-1) + 3\cdot(3-2) = 2+3 = 5$

Note: For any partition $P = a = t_0 < t_1 \cdots < t_n = b$

$$U(f,P) \le \sum_{i=1}^{n} M(f,[a,b])\cdot(t_i - t_{i-1}) = M(f,[a,b])\cdot(b-a)$$

$$L(f,P) \ge \sum_{i=1}^{n} m(f,[a,b])\cdot(t_i - t_{i-1}) = L(f,[a,b])\,(b-a)$$

Thus $\forall$ partition P

$$(*) \quad m(f,[a,b])\cdot(b-a) \le L(f,P) \le U(f,P) \le M(f,[a,b])(b-a)$$

Hence the sets

$\{U(f,P) \mid P \text{ is a partition of } [a,b]\}$

$\{L(f,P) \mid P \text{ is a partition of } [a,b]\}$

are bounded.

We can define $\quad U(f) = \inf\{U(f,P) \mid P \text{ a partition}\}$

$\qquad\qquad\qquad L(f) = \sup\{L(f,P) \mid P \text{ a partition}\}$

Definition A bounded function $f: [a,b] \to \mathbb{R}$ is (Darboux) integrable if $U(f) = L(f)$. In this case we define

$$\int_a^b f(x)\,dx := L(f) = U(f).$$

Remarks We'll prove shortly that $\boxed{L(f) \le U(f)}$

Hence to prove integrability it's enough to show: $U(f) \le L(f)$.

Ex $\quad f: [0,1] \to \mathbb{R} \qquad f(x) = \begin{cases} 1 & x \text{ irrational} \\ 0 & x \text{ rational.} \end{cases}$

Then $\forall\, S \subseteq [0,1] \qquad M(f,S) = 1, \quad m(f,S) = 0$

$\Rightarrow \forall P \qquad L(f,P) = 0, \quad U(f,P) = 1$

$\Rightarrow$ $L(f) = \sup \{L(f,P) \mid P \text{ a partition}\} = 0$

$U(f) = \inf \{U(f,P) \mid P \text{ partition}\} = 1$

f is <u>not</u> (Darboux) integrable.

<u>Ex</u> $f(x) = x$ on $[0,b]$ $(b>0)$. $P = \{0 = t_0 < t_1 < \dots < t_n = b\}$

$$U(f,P) = \sum_{k=1}^{n} M(f, [t_{k-1}, t_k]) \cdot (t_k - t_{k-1}) = \sum_{k=1}^{n} t_k (t_k - t_{k-1})$$

Similarly $L(f,P) = \sum_{k=1}^{n} t_{k-1} (t_k - t_{k-1})$

In particular let P_n be the partition with $t_k = \frac{kb}{n}$.

Then $U(f, P_n) = \sum_{k=1}^{n} \frac{kb}{n} \cdot \left(\frac{kb}{n} - \frac{(k-1)b}{n}\right) = \sum_{k=1}^{n} \frac{kb}{n} \cdot \frac{b}{n} = \frac{b^2}{n^2} \sum_{k=1}^{n} k$

$$= \frac{b^2}{n^2} \frac{n(n+1)}{2} = \frac{b^2}{2} \cdot \frac{n+1}{n}.$$

Similar computation gives $L(f, P_n) = \frac{b^2}{2} \cdot \frac{n+1}{n}$.

Since $U(f) = \inf_P \{U(f,P)\} \le U(f, P_n) = \frac{b^2}{2} \cdot \frac{n+1}{n}$

$$U(f) \le \lim_{n \to \infty} \frac{b^2}{2} \cdot \frac{n+1}{n} = \frac{b^2}{2}$$

Similarly $L(f) = \sup_P \{L(f,P)\} \ge L(f, P_n) = \frac{b^2}{2} \cdot \frac{n-1}{n}$

$$\Rightarrow \quad L(f) \ge \lim_{n \to \infty} \frac{b^2}{2} \frac{n-1}{n} = \frac{b^2}{2}.$$

$$\Rightarrow \quad \frac{b^2}{2} \le L(f) \le U(f) \le \frac{b^2}{2}. \quad \Rightarrow L(f) = U(f) = \frac{b^2}{2}$$

$$\therefore \int_0^b x\,dx = b^2/2.$$

More generally "same" argument gives:

suppose $f : [a,b] \to \mathbb{R}$ is bounded, $\{P_n\}$, $\{Q_n\}$ are sequence of partitions of $[a,b]$ so that the limits

$\lim_{n \to \infty} U(f, P_n)$, $\lim_{n \to \infty} L(f, Q_n)$ exist and are equal.

Then (i) f is integrable and

(ii) $\int_a^b f = \lim_{n \to \infty} L(f, Q_n) = \lim_{n \to \infty} U(f, P_n)$.

We still need to prove: $L(f) \le U(f)$.

Lemma 21.1 Let $f: [a,b] \to \mathbb{R}$ be bounded, P, Q two partitions of $[a,b]$ with $P \subseteq Q$. Then

$$L(f,P) \le L(f,Q) \le U(f,Q) \le U(f,P).$$

Proof We prove that $L(f,P) \le L(f,Q)$. The other proof is similar.

It suffices to consider the case when Q has one more point than P.

That is, $P = \{a = t_0 < t_1 < \dots < t_n = b\}$

$$Q = \{a = t_0 < t_1 < \dots < t_k < u < t_{k+1} < \dots t_n = b\}$$

Then

$$L(f,Q) - L(f,P) = m(f, [t_k, u])(u - t_k) + m(f, [u, t_{k+1}])(t_{k+1} - u) - m(f, [t_k, t_{k+1}])(t_{k+1} - t_k)$$

$$= \big(m(f,[t_k,u]) - m(f,[t_k,t_{k+1}])\big)(u - t_k) + \big(m(f,[u,t_{k+1}]) - m(f,[t_k,t_{k+1}])\big)\cdot(t_{k+1} - u) \ge 0$$

since

$$m(f,[t_k,u]) \ge m(f,[t_k,t_{k+1}])$$
$$m(f,[u,t_{k+1}]) \ge m(f,[t_k,t_{k+1}]).$$

$\square$

Corollary 21.2 Suppose $f: [a,b] \to \mathbb{R}$ is bounded, P, Q two partitions of $[a,b]$. Then

$$L(f,P) \le U(f,Q)$$

Proof $P, Q \subseteq P \cup Q$. Hence by 21.1

$$L(f,P) \le L(f, P \cup Q) \le U(f, P \cup Q) \le U(f,Q). \quad \square$$

Theorem 21.3 $f: [a,b] \to \mathbb{R}$ bounded. Then $L(f) \le U(f)$.

Proof $\forall$ two partitions P, Q of $[a,b]$

$$L(f,P) \le U(f,Q)$$

$$\Rightarrow L(f) := \sup_P L(f,P) \le U(f,Q) \text{ for any } Q$$

$$\Rightarrow \big(L(f) =\big) \sup_P L(f,P) \le \inf_Q U(f,Q) = U(f).$$

$\square$