Last time: $f: [a,b]\to\mathbb{R}$ bounded function, $P=\{a=t_0<t_1<\dots<t_n=b\}$ a partition of $[a,b]$,

$$U(f,P)=\sum_{i=1}^{n} \sup(f|_{[t_i,t_{i-1}]})(t_i-t_{i-1})$$
$$L(f,P)=\sum \inf(f|_{[t_i,t_{i-1}]})(t_i-t_{i-1})$$

$$U(f)=\inf_P U(f,P), \qquad L(f)=\sup_P L(f,P)$$

We proved $\quad L(f)\le U(f) \quad \forall f$

Defined $\quad \int_{[a,b]} f = U(f) \quad$ if $\quad U(f)=L(f)$ and called such f "integrable"

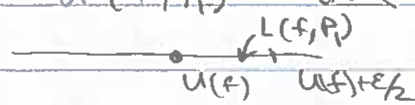
In general, $\quad L(f)\le U(f) \quad$ for bounded f.

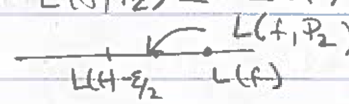
Cauchy criterion for integrability Suppose $f:[a,b]\to\mathbb{R}$ is bounded. Then f is integrable on $[a,b] \iff \forall \varepsilon>0$ $\exists$ a partition P of $[a,b]$ so that

$$U(f,P)-L(f,P)\le\varepsilon.$$

Proof $(\Rightarrow)$ Since f is integrable $U(f)=L(f)$.

Since $U(f)=\inf_P U(f,P)$, $L(f)=\sup_P L(f,P)$ $\exists$ partitions P_1, P_2 s.t. $\quad U(f,P_1)-U(f)\le\varepsilon/2 \quad\quad L(f,P_2)\ge L(f)-\varepsilon/2$.





Let $P=P_1\cup P_2$. Then
$$U(f,P)\le U(f,P_1)\le U(f)+\varepsilon/2$$
$$L(f,P)\ge L(f,P_2)\ge L(f)-\varepsilon/2$$

$\Rightarrow \quad U(f,P)-L(f,P)\le U(f)-L(f)+\varepsilon = \varepsilon \quad$ (since $U(f)=L(f)$.)

$(\Leftarrow)$ Suppose $\forall\varepsilon>0$ $\exists P$ s.t. $\quad U(f)-L(f)\le\varepsilon$.

Then $\quad U(f)\le U(f,P)=U(f,P)-L(f,P)+L(f,P)\le L(f,P)+\varepsilon$
$$\le L(f)+\varepsilon$$

$\Rightarrow \quad \forall\varepsilon>0 \quad (0\le)\, U(f)-L(f)\le\varepsilon, \quad \Rightarrow \quad U(f)=L(f)$ and f is integrable $\square$

To define Riemann integrability we need more definitions.

Def The mesh of a partition $P=t_0<t_1<\dots<t_n$ is

$\mathrm{mesh}(P) = \max \{ |t_i - t_{i-1}| \mid i = 1, \dots n \}$

<u>Def</u> Let $f : [a,b] \to \mathbb{R}$ be a bounded function, $P = \{t_0 = a < t_1 < \dots < t_n = b\}$ a partition of $[a,b]$. Choose $x_k \in [t_{k-1}, t_k]$ $\forall k$.
The corresponding <u>Riemann sum</u> is

$$S := \sum_{k=1}^{n} f(x_k) \cdot (t_k - t_{k-1})$$

<u>Def</u> A (bounded) function $f : [a,b] \to \mathbb{R}$ is <u>Riemann integrable</u> if $\exists R \in \mathbb{R}$ so that $\forall \varepsilon > 0$ $\exists \delta > 0$ with the property that $\forall$ partition P with $\mathrm{mesh}(P) < \delta$ $\forall$ Riemann sum S of f associated with P

$$|S - R| < \varepsilon.$$

R is called the Riemann integral of f.

<u>Theorem 22.1</u> A bounded function $f : [a,b] \to \mathbb{R}$ is Riemann integrable $\Leftrightarrow$ f is Darboux integrable. The values of the two integrals agree.

<u>Proof</u> postponed indefinitely.

<u>Properties of integrals</u>

<u>Theorem 22.2</u> Every monotonic function $f : [a,b] \to \mathbb{R}$ is integrable
<u>Proof</u> We consider the case where f is (weakly) increasing:
$\forall x_1, x_2 \in [a,b]$ with $x_1 < x_2$ $\quad f(x_1) \le f(x_2)$
Since $f(a) \le f(x) \le f(b)$ $\quad \forall x \in [a,b]$, f is bounded
We argue: $\forall \varepsilon > 0$ $\exists$ a partition P of $[a,b]$ s.t.

$$U(f,P) - L(f,P) < \varepsilon$$

Let $P_n = t_0 = a < t_1 < \dots < t_n = b$ where $t_k := a + \frac{b-a}{n} \cdot k$

Then $U(f, P_n) = \sum \sup (f|_{[t_k, t_{k-1}]}) \cdot (t_k - t_{k-1}) = \sum_{k=1}^{n} f(t_k) \cdot \frac{b-a}{n}$.

Similarly, $U(f, P_n) = \sum_{k=1}^{n} f(t_{k-1}) \cdot \frac{b-a}{n}$.
Hence
$$U(f, P_n) - L(f, P_n) = \frac{b-a}{n} \sum_{k=1}^{n} \left(f(t_k) - f(t_{k-1}) \right)$$
$$= \frac{b-a}{n} \left((f(t_1) - f(t_0)) + (f(t_2) - f(t_1)) + \ldots + (f(t_n) - f(t_{n-1})) \right)$$
$$= \frac{b-a}{n} \cdot (f(b) - f(a)).$$

Now given $\varepsilon > 0$ choose n st $\frac{(b-a)(f(b)-f(a))}{n} < \varepsilon$

Then $U(f, P_n) - L(f, P_n) < \varepsilon$ and we're done by Cauchy criterion for integrability $\square$

<u>Theorem 22.3</u> Every continuous function $f: [a,b] \to \mathbb{R}$ is integrable.

<u>Proof</u> Since $[a,b]$ is compact, $f: [a,b] \to \mathbb{R}$ is uniformly continuous: $\forall \varepsilon > 0$ $\exists \delta > 0$ so that
$$|x-y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{b-a} \quad \forall x, y \in [a,b]$$
Let P be a partition of $[a,b]$ with $\mathrm{mesh}(P) < \delta$.

$P = \{t_0 < t_1 < \ldots < t_n\}$

$\forall k$ $\exists x_k, y_k \in [t_k, t_{k-1}]$ st $f(x_k) = \sup\{f(x) \mid x \in [t_k, t_{k-1}]\}$
$f(y_k) = \inf\{f(x) \mid x \in [t_k, t_{k-1}]\}$

Then $M(f, [t_k, t_{k-1}]) - m(f, [t_k, t_{k-1}]) = f(x_k) - f(y_k) < \frac{\varepsilon}{b-a}$
since $|x_k - y_k| \leq t_k - t_{k-1} < \delta$

Therefore
$$U(f,P) - L(f,P) = \sum_k \left(M(f, [t_k, t_{k-1}]) - m(f, [t_k, t_{k-1}]) \right) \cdot (t_k - t_{k-1})$$
$$\leq \sum_k \frac{\varepsilon}{b-a} \cdot (t_k - t_{k-1}) = \frac{\varepsilon}{b-a} \cdot (b-a) = \varepsilon$$

Cauchy criterion now implies that f is integrable $\square$

<u>Theorem 22.4</u> Suppose $f, g : [a,b] \to \mathbb{R}$ are bounded and integrable

(i) $\forall c \in \mathbb{R}$ cf is integrable and $\int_a^b cf = c \int_a^b f$

(ii) $f+g$ is integrable and $\int_a^b (f+g) = \int_a^b f + \int_a^b g$.

<u>Remark</u> Theorem says: (a) bounded integrable functions form a vector space and (b): $\int_a^b$: integrable functions $\to \mathbb{R}$, $f \mapsto \int_a^b f$ is <u>linear</u>.

<u>Proof</u> It's enough to prove (i) for $c>0$ and for $c=-1$:

If $c<0$, $c = (-1)|c|$ and then $\int_a^b cf = \int_a^b (-1)|c| f = (-1) \int_a^b |c| f$

$= (-1)|c| \int_a^b f = c \int_a^b f$.

If $c=0$, $cf = 0$ and then $\int_a^b 0 \cdot f = 0 = 0 \cdot \int_a^b f$.

<u>$c=-1$</u> $\sup(-f|_{[t_k, t_{k-1}]}) = -\inf(f|_{[t_k, t_{k-1}]})$

$\inf(-f|_{[t_k, t_{k-1}]}) = -\sup(f|_{[t_k, t_{k-1}]})$ $\forall [t_k, t_{k-1}] \subseteq [a,b]$

$\Rightarrow$ $\forall$ partition P of $[a,b]$

$U(-f, P) = -L(f, P)$

$L(-f, P) = -U(f, P)$

$\Rightarrow U(-f) = \inf_P U(-f, P) = \inf_P (-L(f,P)) = -\sup_P L(f,P) = -L(f)$.

Similarly $L(-f) = -U(f)$.

$\Rightarrow$ $-f$ is integrable and $\int_a^b (-f) = -\int_a^b f$.

<u>$c>0$</u> $\sup(cf(x)|_{[t_k, t_{k-1}]}) = c \sup(f(x)|_{[t_k, t_{k-1}]})$

$\inf(cf(x)|_{[t_k, t_{k-1}]}) = c \inf(f(x)|_{[t_k, t_{k-1}]})$

$\Rightarrow U(cf, P) = c \cdot U(f, P)$, $L(cf, P) = c\, L(f, P)$ $\forall P$

$\Rightarrow U(cf) = c\, U(f) = c\, L(f) = L(cf)$,

$\Rightarrow$ cf is integrable and $\int_a^b cf = c \int_a^b f$. $\square$

Proof of (ii) next time ...