MATH 424 Eugene Lerman

we haven't finished proving:

Thm 22.4 $f, g : [a,b] \to \mathbb{R}$ (bounded and) integrable. Then

(i) $\forall c \in \mathbb{R}$ cf is integrable and $\int_{[a,b]} cf = c \int_{[a,b]} f$

(ii) $f+g$ is integrable and

$$\int_{[a,b]} (f+g) = \left(\int_{[a,b]} f\right) + \left(\int_{[a,b]} g\right).$$

Proof We observed last time that for (i) it's enough to consider two special cases: $c = -1$ and $c > 0$.

We observed: $\forall$ partition P of $[a,b]$

$$U(-f, P) = -L(f, P), \quad L(-f, P) = -U(f, P)$$

Hence

$$U(-f) = \inf_P (U(-f,P)) = \inf_P (-L(f,P)) = -\sup_P (L(f,P)) = -L(f)$$

Similarly

$$L(-f) = -U(f).$$

Since f is integrable, $U(f) = L(f)$. $\Rightarrow$ $U(-f) = L(-f)$ and $-f$ is integrable.

$c > 0$ Observe that

$$\sup\{cf(x) \mid x \in [t_{k-1}, t_k]\} = c \sup\{f(x) \mid x \in [t_{k-1}, t_k]\}$$
$$\inf\{cf(x) \mid x \in [t_{k-1}, t_k]\} = c \inf\{f(x) \mid x \in [t_{k-1}, t_k]\}.$$

$\Rightarrow$ $U(cf, P) = c\, U(f,P)$ $\forall P$

$L(cf, P) = c\, L(f, P)$ $\forall P$

$\Rightarrow$ $U(cf) = \inf_P U(cf,P) = \inf_P (c\, U(f,P)) = c \inf_P (U(f,P)) = c\, U(f)$

Similarly $L(cf) = c\, L(f)$.

Since $U(f) = L(f) = \int_{[a,b]} f$ it follows that

$U(c f) = c \int_{[a,b]} f = L(c f)$. and (i) follows.

(ii) Since f, g are integrable $\forall \varepsilon > 0$ $\exists$ partitions P_1, P_2 of $[a,b]$ s.t.

(1) $U(f, P_1) - L(f, P_1) < \varepsilon/2$

(2) $U(g, P_2) - L(g, P_2) < \varepsilon/2$

Let $P = P_1 \cup P_2$. Then $U(f,P) - L(f,P) < \varepsilon/2$
$$U(g,P) - L(g,P) < \varepsilon/2.$$

Next note that $\forall S \subseteq [a,b]$, $\forall x \in S$

$$(f+g)(x) = f(x) + g(x) \leq \sup\{f(y) \mid y \in S\} + \sup\{g(y) \mid y \in S\}$$

$$\Rightarrow \sup\{(f+g)(x) \mid x \in S\} \leq \sup_{y \in S} f(y) + \sup_{y \in S} g(y)$$

Similarly $\inf_{x \in S} (f+g)(x) \geq \inf_{x \in S} f(x) + \inf_{x \in S} g(x)$

$$\Rightarrow L(f+g, P) \geq L(f,P) + L(g,P)$$

$$U(f+g,P) \leq U(f,P) + U(g,P)$$

$$\Rightarrow U(f+g,P) - L(f+g,P) \leq (U(f,P) + U(g,P)) - (L(f,P) + L(g,P))$$

$$= (U(f,P) - L(g,P)) + (U(g,P) - L(g,P))$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Done by Cauchy's criterion. □

Theorem 23.1 Suppose $a < b < c$, $f : [a,c] \to \mathbb{R}$ is bounded and $f|_{[a,b]}$, $f|_{[b,c]}$ are integrable. Then f is integrable and

$$\int_{[a,c]} f = \int_{[a,b]} f + \int_{[b,c]} f.$$

Proof Given $\varepsilon > 0$ $\exists$ partitions P_1 of $[a,b]$, P_2 of $[b,c]$ so that $U(f,P_1) - L(f,P_1) < \varepsilon/2$ and $U(f,P_2) - L(f,P_2) < \varepsilon/2$

Let $P = P_1 \cup P_2$.

Then $U(f,P) = U(f,P_1) + U(f,P_2)$, $L(f,P) = L(f,P_1) + L(f,P_2)$.

Hence $U(f,P) - L(f,P) < \varepsilon/2 + \varepsilon/2 = \varepsilon$

$\Rightarrow$ f is integrable on $[a,c]$.

Furthermore

$$\int_{[a,c]} f = U(f) \leq U(f,P) = U(f,P_1) + U(f,P_2)$$

$$< L(f,P_1) + \varepsilon/2 + L(f,P_2) + \varepsilon/2 \leq \int_{[a,b]} f + \int_{[b,c]} f + \varepsilon.$$

Similarly $\int_{[a,c]} f \geq L(f,P) = L(f,P_1) + L(f,P_2) > (U(f,P_1) - \varepsilon/2) + (U(f,P_2) - \varepsilon/2)$

$$\geq \left(\int_{[a,b]} f - \varepsilon/2\right) + \left(\int_{[b,c]} f - \varepsilon/2\right) = \left(\int_{[a,b]} f + \int_{[c,d]} f\right) - \varepsilon$$

$$\Rightarrow \quad \left| \int_{[a,c]} f - \left(\int_{[a,b]} f + \int_{[b,c]} f\right) \right| < \varepsilon \quad \forall \varepsilon.$$

$$\therefore \quad \int_{[a,c]} f = \int_{[a,b]} f + \int_{[b,c]} f.$$

Remark At this point, given $f : [a,b] \to \mathbb{R}$, integrable, we can define (*) $\int_b^a f(x)\,dx = -\int_{[a,b]} f$

And then $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

even if b is not between a and c.

((provided f is integrable on the three relevant intervals).

WARNING

$\int_a^b f(x)\,dx$ is then not an integral of a function;

it's the integral of the 1-form $f(x)\,dx$,

it's an oriented integral. Forms are discussed in MATH 425.

Theorem 23.2 Suppose $f : [a,b] \to [c,d]$ is integrable, $g : [c,d] \to \mathbb{R}$ continuous. Then $h(x) := g(f(x))$ is integrable on $[a,b]$.

Proof postponed

Theorem 23.3 if $f, g : [a,b] \to \mathbb{R}$ are integrable and $f(x) \leq g(x)$ $\forall x$

Then $\int_{[a,b]} f \leq \int_{[a,b]} g$

Proof For any partition P, $U(f,P) \leq U(g,P)$

Hence ~~U(~~ $U(f) \leq \inf_{P'} U(f,P') \leq U(f,P) \leq U(g,P) \quad \forall P$

$$\Rightarrow \int_{[a,b]} f = \inf_P U(f,P) \leq \inf_P U(g,P) = \int_{[a,b]} g.$$

Corollary 23.4 If $f:[a,b]\to\mathbb{R}$ is integrable, so is $|f|$.
Moreover $$\left|\int_{[a,b]} f\right| \leq \int_{[a,b]} |f|.$$

Proof Since $g(y) = |y|$ is continuous, $|f|(x) = (g\circ f)(x)$ is integrable by 23.2.

Since $$-|f(x)| \leq f(x) \leq |f(x)| \quad \forall x$$
$$-|f| \leq f \leq |f|.$$

By 23.3
$$-\int_{[a,b]} |f| \leq \int_{[a,b]} f \leq \int_{[a,b]} |f|$$

$$\Rightarrow \left|\int_{[a,b]} f\right| \leq \int_{[a,b]} |f|.$$

$\square$

Cor 23.5 (i) $\forall$ integrable function $q:[a,b]\to\mathbb{R}$, q^2 is integrable

(ii) If $f,g:[a,b]\to\mathbb{R}$ are integrable, then so is $f\cdot g$.

Proof

(i) Since $h(y) = y^2$ is continuous, $q^2 = h\circ q$ is integrable

(ii) $$f\cdot g = \tfrac{1}{4}\left((f+g)^2 - (f-g)^2\right)$$

f,g integrable $\Rightarrow$ $f\pm g$ are integrable $\Rightarrow$ $(f+g)^2$, $(f-g)^2$ are integrable $\Rightarrow$ $f\cdot g$ is integrable.

$\square$