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We need to prove

- **Thm 23.2** $f:[a,b]\to[c,d]$ integrable, $g:[c,d]\to\mathbb{R}$ continuous. Then $h:=g\circ f:[a,b]\to\mathbb{R}$ is integrable.

Proof By Cauchy's criterion it's enough to show: $\forall\varepsilon>0$ $\exists$ partition $P=\{t_0<\dots<t_n\}$ of $[a,b]$ so that

$$(*)\qquad U(h)-L(h)<\varepsilon$$

1. Let $K=\sup\{|g(y)|\mid y\in[c,d]\}$
2. Choose any ε' so that $0<\varepsilon'<\dfrac{\varepsilon}{2K+(b-a)}$
3. Since $[c,d]$ is compact, $g:[c,d]\to\mathbb{R}$ is uniformly continuous.
   $\Rightarrow\exists\,\delta$ so that $\delta<\varepsilon'$ and $|s-t|<\delta\Rightarrow|g(s)-g(t)|<\varepsilon'$.
4. Since f is integrable $\exists$ a partition $P=\{a=t_0<t_1<\dots<t_n=b\}$ of $[a,b]$ s.t.
   $$U(f,P)-L(f,P)<\delta^2.$$

We now argue that P is the desired partition

Notation For any $\varphi:[a,b]\to\mathbb{R}$, ~~see~~ bounded, set

$$M_i(\varphi)=\sup\{\varphi|_{[t_{i-1},t_i]}\}\qquad m_i(\varphi)=\inf(\varphi|_{[t_{i-1},t_i]})$$

Let

$$A=\{k\in\{1,\dots n\}\mid M_k(f)-m_k(f)<\delta\}$$
$$B=\{k\in\{1,\dots n\}\mid M_k(f)-m_k(f)\geq\delta\}.$$

Then $\forall\, i\in A$ $\forall\, x,y\in[t_{i-1},t_i]$

$$|f(x)-f(y)|\leq M_i(f)-m_i(f)<\delta$$

And then

$$|g(f(x))-g(f(y))|<\varepsilon'$$
$$\Rightarrow\quad M_i(g\circ f)-m_i(g\circ f)\leq\varepsilon'$$
$$\Rightarrow\sum_{i\in A}(M_i(g\circ f)-m_i(g\circ f))(t_i-t_{i-1})\leq\varepsilon'\sum_{i\in A}(t_i-t_{i-1})\leq\varepsilon'(b-a)$$

If $i\in B$ then $\frac{1}{\delta}(M_i(f)-m_i(f))\geq 1$

$$\sum_{i\in B}(t_i-t_{i-1})\leq\sum_{i\in B}\frac{1}{\delta}(M_i(f)-m_i(f))(t_i-t_{i-1})\leq\frac{1}{\delta}(U(f,P)-L(f,P))$$

$\leq \frac{1}{\delta} \cdot \delta^2 = \delta < \varepsilon'.$

Since $M_i(g\circ f) - m_i(g\circ f) \leq 2K \ \forall i$

$$\sum_{i\in B} (M_i(g\circ f) - m_i(g\circ f))(t_i - t_{i-1}) \leq 2K \sum_{i\in B} (t_i - t_{i-1}) < 2K\varepsilon'.$$

Hence

$$U(g\circ f, P) - L(g\circ f, P) = \sum_{i\in A} (M_i(g\circ f) - m_i(g\circ f))(t_i - t_{i-1})$$

$$+ \sum_{i\in B} (M_i(g\circ f) - m_i(g\circ f))(t_i - t_{i-1}) < \varepsilon'(b-a) + 2K\varepsilon'$$

$$= \varepsilon'((b-a)+2K) < \varepsilon.$$

$\square$

<u>Theorem 24.1</u> (Fundamental theorem of calculus, version 1). Suppose $g : [a,b] \to \mathbb{R}$ continuous, $g|_{(a,b)}$ is differentiable, and g' is (bounded and) integrable on $[a,b]$. Then

$$\int_{[a,b]} g' \left(= \int_a^b g'(x)\,dx \right) = g(b) - g(a).$$

<u>Proof</u> Fix a partition $P = \{a = t_0 < t_1 < \dots < t_{n-1} < t_n = b\}$.

By Mean Value Theorem $\forall k \quad \exists x_k \in [t_k, t_{k-1}]$ so that

$$g'(x_k) = \frac{g(t_k) - g(t_{k-1})}{t_k - t_{k-1}}$$

ie. $$g'(x_k)(t_k - t_{k-1}) = g(t_k) - g(t_{k-1}).$$

$\Rightarrow \forall k$

$$m(g', [t_{k-1}, t_k])(t_k - t_{k-1}) \leq \underbrace{g'(x_k)(t_k - t_{k-1})}_{g(t_k) - g(t_{k-1})} \leq M(g', [t_{k-1}, t_k]) \cdot (t_k - t_{k-1})$$

Hence

$$L(g', P) = \sum_k m(g', [t_{k-1}, t_k])(t_k - t_{k-1}) \leq \sum_{k=1}^n (g(t_k) - g(t_{k-1})) = g(b) - g(a)$$

Similarly

$$U(g', P) = \sum_k M(g', [t_{k-1}, t_k])(t_k - t_{k-1}) \geq \sum_{k=1}^n (g(t_k) - g(t_{k-1})) = g(b) - g(a)$$

Hence
$$\int_{[a,b]} g' = \sup_{P} L(g',P) \le g(b)-g(a)$$
$$\int_{[a,b]} g' = \inf_{P} U(g',P) \ge g(b)-g(a)$$
Therefore
$$\int_{[a,b]} g' = g(b)-g(a).$$
□

Corollary 24.2 (integration by parts). Suppose $f:[a,b]\to\mathbb{R}$ are continuous, differentiable on (a,b) and f', g' are integrable on $[a,b]$. Then
$$\int_{[a,b]} f\cdot g' = (f(b)g(b)-f(a)g(a)) - \int_{[a,b]} f'g$$

Proof $(fg)' = f'\cdot g + f\cdot g'$
Since f', g', f, g are integrable so is $(fg)'$.
By FTC v.1
$$f(b)g(b)-f(a)g(a) = \int_{[a,b]} (fg)' = \int_{[a,b]} (f'g+fg')$$
$$= \int_{[a,b]} f'\cdot g + \int_{[a,b]} f\cdot g'.$$

Theorem 24.3 (Fundamental theorem of calculus v.2)
Suppose $f:[a,b]\to\mathbb{R}$ is (bounded and) integrable. Then
$$F(x) := \int_a^x f(u)\,du \quad \text{is continuous}$$
Moreover if f is continuous at $x_0\in(a,b)$ then F is differentiable at x_0 and
$$F'(x_0) = f(x_0).$$

Proof Since f is bounded $\exists M>0$ s.t $|f(x)|<M \ \forall x\in[a,b]$.
We argue first that F is uniformly continuous.

For any $x, y \in [a,b)$, $x < y$

$$|F(y) - F(x)| = \left|\int_a^y F - \int_a^x F\right| = \left|\int_a^x F + \int_x^y F - \int_a^x F\right|$$

$$= \left|\int_{[x,y]} F\right| \le \int_{[x,y]} |F| \le \int_{[x,y]} M = M(y-x) = M|y-x|.$$

$\therefore$ F is (uniformly) continuous

We now argue that F is differentiable at x_0 and that $F'(x_0) = f(x_0)$, ie

$$\lim_{h \to 0} \left|\frac{1}{h}\left(F(x_0+h) - F(x_0)\right) - f(x_0)\right| = 0$$

Note: $f(x_0) = \frac{1}{h} \cdot f(x_0) \cdot ((x_0+h) - x_0) = \frac{1}{h} \int_{x_0}^{x_0+h} f(x_0)\,dx$

$$\Rightarrow \left|\frac{1}{h}\left(F(x_0+h) - F(x_0)\right) - f(x_0)\right| = \left|\frac{1}{h}\left(\int_a^{x_0+h} f - \int_a^{x_0} f\right) - f(x_0)\right|$$

$$= \left|\frac{1}{h}\left(\int_{x_0}^{x_0+h} f - \frac{1}{h}\int_{x_0}^{x_0+h} f(x_0)\right)\right| = \left|\frac{1}{h}\int_{x_0}^{x_0+h} \left(f(u) - f(x_0)\right) du\right|$$

$$\le \left|\int_{x_0}^{x_0+h} \frac{|f(u) - f(x_0)|}{|h|}\,du\right| \le \sup\{|f(u) - f(x_0)| \mid u \in [x_0 - |h|, x_0 + |h|]\}$$

Since f is continuous at x_0, $\forall \varepsilon > 0$ $\exists \delta > 0$ st

$$|u - x_0| < \delta \Rightarrow |f(u) - f(x_0)| < \varepsilon/2$$

Therefore if $|h - 0| < \delta$ then

$$\left|\frac{1}{h}\left(F(x_0+h) - F(x_0)\right) - f(x_0)\right| \le \varepsilon/2 < \varepsilon.$$

$$\Rightarrow \lim_{h \to 0} \left|\frac{1}{h}\left(F(x_0+h) - F(x_0)\right) - f(x_0)\right| = 0.$$