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Last time:

23.2 $f:[a,b]\to[c,d]$ integrable, $g:[c,d]\to\mathbb{R}$ continuous, Then $g\circ f$ is integrable.

Fact $f(x)=\begin{cases}1/q & \text{if } x=p/q \ \gcd(p,q)=1\\ 0 & x \text{ irrational}\end{cases}$ is integrable on any $[a,b]$

$g:\mathbb{R}\to\mathbb{R}$ $\quad g(x)=\begin{cases}1 & x>0\\ 0 & x\le 0\end{cases}$ is integrable on any interval

But $g\circ f\,(x)=\begin{cases}1 & x \text{ rational}\\ 0 & x \text{ irrational}\end{cases}$ is not integrable

Moral in 23.2 we need g to be continuous on all of $[c,d]$.

24.1 $g:[a,b]\to\mathbb{R}$ continuous, g' exists on (a,b) and is integrable on $[a,b]$

Then $\int_a^b g'(u)\,du = g(b)-g(a)$

24.3 $f:[a,b]\to\mathbb{R}$ integrable $\Rightarrow F(x)=\int_a^x f(u)\,du$ is continuous.

If f is continuous at $x_0\in(a,b)$, then F is differentiable at x_0 and $F'(x_0)=f(x_0)$

"Ex" $f(x)=1/x$ is continuous on $(0,\infty)$.

$\Rightarrow F(x):=\int_1^x \frac{1}{u}\,du$ is differentiable on $(0,\infty)$.

Moreover $F'(x)=1/x>0 \Rightarrow F:(0,\infty)\to\mathbb{R}$ is strictly increasing

The book defines $\ln(x):=F(x)=\int_1^x \frac{1}{u}\,du$

Since $F(x)$ is strictly increasing and $F'(x)\neq 0$, it has a differentiable inverse. We define $\exp(y)=F^{-1}(y)$.

Since

$$\frac{d}{dy}(F^{-1}(y))=\frac{1}{F'(F^{-1}(y))} \quad \text{and since } F'(x)=1/x$$

$$\frac{d}{dy}(\exp(y))=\frac{1}{1/\exp(y)}=\exp(y).$$

Since $\ln(1)=\int_1^1 \frac{du}{u}=0$, $\exp(0)=1$.

<u>Lemma 25.1</u> (i) $\ln(xy) = \ln(x) + \ln(y)$

(ii) $\ln(\frac{1}{y}) = -\ln(y)$

(iii) $\ln(x^n) = n\ln(x) \quad \forall n \in \mathbb{Z}$.

<u>Proof</u> (i) By the chain rule

$$\frac{d}{dx}\left(\ln(xy) - \ln x\right) = \frac{1}{xy}\cdot\frac{d}{dx}(xy) - \frac{1}{x} = \frac{1}{xy}\cdot y - \frac{1}{x} = 0.$$

$\Rightarrow \ln(xy) - \ln x = c$ for some $c \in \mathbb{R}$.

Since $\ln(1) = 0$, $c = \ln(1\cdot y) - \ln(1) = \ln(y)$.

$\therefore \ln(xy) - \ln(x) = \ln y$ and (i) follows.

(ii) $0 = \ln 1 = \ln(y\cdot\frac{1}{y}) = \ln y + \ln(\frac{1}{y}) \Rightarrow \ln(1/y) = -\ln(y)$.

(iii) For $n > 0$, (iii) follows from (i) by induction.

if $n < 0$, $\ln(x^n) = \ln((x^{-1})^{-n}) = (-n)\ln(\frac{1}{x}) = (-n)\cdot(-1)\ln x = n\ln x$

<u>Note</u> Since $\ln(2) > \ln(1) = 0$, $\ln(2^n) = n\ln 2 \xrightarrow[n\to\infty]{} +\infty$

$\ln : [1,\infty) \to [0,\infty)$ is onto by the intermediate value theorem.

Since $\ln(\frac{1}{x}) = -\ln x$, $\ln : (0,1] \to (-\infty, 0]$ is also onto

$\therefore \ln : (0,\infty) \to \mathbb{R}$ is onto

$\therefore$ domain of $\exp = (\ln)^{-1}$ is all of $\mathbb{R}$.

<u>Note</u> $\ln(\exp(x)\exp(y)) = \ln(\exp x) + \ln(\exp(y)) = x+y$

$\Rightarrow \exp(x)\cdot\exp(y) = \exp(x+y)$

$\Rightarrow \exp(-x)\exp(x) = \exp(-x+x) = \exp(0) = 1. \Rightarrow \exp(-x) = \frac{1}{\exp(x)}$.

<u>Def.</u> For $x > 0$ we define $x^\alpha : (0,\infty) \to \mathbb{R}$ by

$$\boxed{x^\alpha = \exp(\alpha\ln(x))}$$

<u>Note:</u> For $\alpha = n \in \mathbb{N}$: $x^\alpha = \exp(n\ln(x)) = \exp(\ln(x^n)) = x^n$

So this new definition agrees with the older one

if α is a natural number.

Theorem 25.2
i) $x^\alpha \cdot x^\beta = x^{\alpha+\beta}$
ii) $\frac{x^\alpha}{x^\beta} = x^{\alpha-\beta}$
iii) $(x^\alpha)^\beta = x^{\alpha\beta}$
(iv) $(xy)^\alpha = x^\alpha y^\alpha$ (v) $\frac{d}{dx}(x^\alpha) = \alpha x^{\alpha-1}$
(iv) $\frac{d}{d\alpha} x^\alpha = \ln x \cdot x^\alpha$

Proof (i) $x^\alpha x^\beta = \exp(\alpha \ln x) \exp(\beta \ln x) = \exp((\alpha+\beta)\ln(x)) = x^{\alpha+\beta}$

(ii) $\frac{1}{x^\beta} = \frac{1}{\exp(\beta \ln(x))} = \exp(-\beta \ln x) = x^{-\beta}$

(iii)

$$(x^\alpha)^\beta = (\exp(\alpha \ln(x)))^\beta = \exp[\beta \underbrace{\ln}_{\text{id}} (\exp(\alpha \ln(x)))]$$
$$= \exp(\beta \cdot (\alpha \ln x)) = x^{\alpha\beta}$$

(iv)

$$(xy)^\alpha = \exp(\alpha \ln(xy)) = \exp(\alpha \ln x + \alpha \ln y)$$
$$= \exp(\alpha \ln x) \cdot \exp(\alpha \ln y) = x^\alpha \cdot y^\alpha.$$

(v) $\frac{d}{dx}(x^\alpha) = \frac{d}{dx} \exp(\alpha \ln x) = \exp(\alpha \ln x) \cdot \frac{d}{dx}(\alpha \ln x) = x^\alpha \cdot \alpha \cdot \frac{1}{x} = \alpha x^{\alpha-1}.$

(vi) $\frac{d}{d\alpha}(x^\alpha) = \frac{d}{d\alpha} \exp(\alpha \ln x) = \exp(\alpha \ln x) \cdot (\ln x) = \ln x \cdot x^\alpha.$

Remark Define $e := \exp(x)$. Then

$$e^\alpha = \exp(\alpha \ln(e)) = \exp(\alpha) \quad \forall \alpha, \text{ ie}$$
$$\exp(x) = e^x.$$

Lemma 25.3 $\lim_{n\to\infty} (1+\frac{1}{n})^n = e$.

Proof $(1+1/n)^n = \exp(n \ln(1+1/n))$. Since $\exp(x)$ is continuous

$$\Rightarrow \lim_{n\to\infty} ((1+1/n)^n = \exp(\lim_{n\to\infty}(n \ln(1+1/n)))$$

$$\lim_{n\to\infty} n \ln(1+1/n) = \lim_{n\to\infty} \frac{\ln(1+1/n)}{1/n} = \lim_{h\to 0} \frac{\ln(1+h) - \overbrace{\ln(1)}^{=0}}{h} = \left(\frac{d}{dx}\ln(x)\right)\Big|_{x=1} = \frac{1}{1}$$

$$\Rightarrow \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e^{\lim_{n\to\infty} n\ln(1+1/n)} = e^1 = e.$$

□

Thm 25.4 (Change of variables) Suppose I, J are open intervals $u: I \to J$ is differentiable and u' is continuous (i.e. $u \in C^1(I)$)

Suppose $f: J \to \mathbb{R}$ is continuous. Then $\forall\, a, b \in I$

$$\int_a^b (f\circ u)(x)\, u'(x)\,dx = \int_{u(a)}^{u(b)} f(u)\,du.$$

Proof Since $f\circ u$, u' are continuous, $(f\circ u)\cdot u'$ is continuous on I, hence integrable. Fix $c \in J$ and define $F(y) = \int_c^y f(t)\,dt$

Then by the fund theorem of calculus, v2, F is differentiable and $F'(y) = f(y)$ $\forall\, y \in J$.

Let $g := F\circ u$.

By the chain rule

$$g'(x) = F'(u(x))\cdot u'(x) = f(u(x))\cdot u'(x)$$

$$\Rightarrow \int_a^b (f\circ u)(x)\cdot u'(x)\,dx = \int_a^b g'(x)\,dx \underset{\text{Fund. Thm, v1,}}{=} g(b) - g(a)$$

$$= F(u(b)) - F(u(a)) = \int_c^{u(b)} f(t)\,dt - \int_c^{u(a)} f(t)\,dt$$

$$= \int_{u(a)}^{u(b)} f(t)\,dt.$$

□