MATH 424 Eugene Lerman 03/20/2024

Last time: • proved change of variables formula for integrals:

$$\int_a^b f(u(x))u'(x)\,dx = \int_{u(a)}^{u(b)} f(y)\,dy$$

• Defined $\ln : (0,\infty) \to \mathbb{R}$ by $\ln(x) = \int_1^x \frac{1}{u}\,du$.

Proved $\ln : (0,\infty) \to \mathbb{R}$ is a differentiable bijection. Defined $\exp : \mathbb{R} \to (0,\infty)$ by $\exp = (\ln)^{-1}$.

Defined $e := \exp(1)$

$x^\alpha := \exp(\alpha \ln(x)) \quad \forall\, x > 0\ \forall \alpha$

Then: $e^x = \exp(x) \quad \forall x \in \mathbb{R}$.

Lemma 25.3 $\lim_{n\to\infty} (1+\frac{1}{n})^n = e$.

Proof $(1+\frac{1}{n})^n = \exp(n \ln(1+1/n))$. Since $\exp$ is continuous

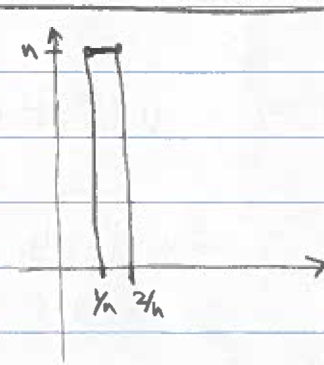
$$\lim_{n\to\infty}(1+\tfrac{1}{n})^n = \exp\left(\lim_{n\to\infty}(n \ln(1+1/n))\right)$$

$$\lim_{n\to\infty} n\ln(1+1/n) = \lim_{n\to\infty}\frac{\ln(1+1/n)}{1/n} = \lim_{h\to 0}\frac{\ln(1+h)}{h} = \lim_{h\to 0}\frac{\ln(1+h)-\ln(1)}{h}$$

$$= \frac{d}{dx}(\ln(x))\Big|_{x=1} = \frac{1}{1}$$

$$\Rightarrow \lim_{n\to\infty}(1+1/n)^n = \exp(1) = e.$$

Chapter VII Interchange of limit operations.



Ex $f_n : [0,2] \to \mathbb{R}$ $f_n(x) = \begin{cases} n & \frac{1}{n} \le x \le \frac{2}{n} \\ 0 & \text{otherwise} \end{cases}$

$\int_0^2 f_n(x)\,dx = n\cdot\left(\frac{2}{n} - 1/n\right) = 1$

So $\lim_{n\to\infty} \int_0^2 f_n(x)\,dx = 1$

But $f_n(x) \to 0 \quad \forall x \in [0,2]$. So $\lim_{n\to\infty}\int_{[0,2]} f_n \neq \int_{[0,2]}(\lim f_n)$

Ex $\exists$ a sequence of integrable function $f_n : [0,1] \to \mathbb{R}$
s.t. $f(x) = \lim_{n\to\infty} f_n(x)$ is not integrable.

Construction

$\mathbb{Q} \cap [0,1]$ is countable, so $\exists$ a bijection $\mathbb{N} \to \mathbb{Q} \cap [0,1]$, $n \mapsto r_n$.
Now define $f_n : [0,1] \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} 1 & x = r_1, \dots r_n \\ 0 & \text{otherwise} \end{cases}.$$

Each f_n is integrable. But $\lim_{n\to\infty} f_n(x) = \begin{cases} 1 & x \in \mathbb{Q} \cap [0,1] \\ 0 & x \notin \mathbb{Q} \cap [0,1] \end{cases}$
which is not integrable.

Theorem 26.1 Suppose $\{f_n : [a,b] \to \mathbb{R}\}$ is a sequence integrable functions and suppose $f_n \to f$ uniformly. Then f is integrable and $\int_{[a,b]} f = \lim_{n\to\infty} \int_{[a,b]} f.$

Recall $f_n \to f$ uniformly on $[a,b]$ if $\forall \varepsilon > 0$ $\exists N$

$$(*) \quad \sup_{x \in [a,b]} |f_n(x) - f(x)| < \varepsilon \qquad \forall n \geq N.$$

Proof of 26.1 Pick a partition P of $[a,b]$ so that

$$U(f_N, P) - L(f_N, P) < \varepsilon$$

Then, $\forall i$

$$\sup_{[t_i, t_{i+1}]} f \leq \sup_{[t_i, t_{i+1}]} (f_N - f) + \sup_{[t_i, t_{i+1}]} f_N$$

$$\Rightarrow \quad U(f, P) \leq U(f - f_N, P) + U(f_N, P)$$
$$\leq \varepsilon \cdot (b-a) + U(f_N, P)$$

Similarly

$$L(f, P) \geq L(f_N, P) + (-\varepsilon) \cdot (b-a)$$

Hence

$$U(f,P) - L(f,P) \leq 2\varepsilon \cdot (b-a) + \varepsilon$$
$$\Rightarrow f \text{ is integrable.}$$

Finally $\left| \int_{[a,b]} f - \int_{[a,b]} f_n \right| \le \int_{[a,b]} |f - f_n|$

$$\le \left(\sup_{[a,b]} |f(x) - f_n(x)| \right) \cdot (b-a) \xrightarrow[n\to\infty]{} 0$$

$$\therefore \lim_{n\to\infty} \int_{[a,b]} f_n = \int_{[a,b]} f \quad \left(= \int_{[a,b]} (\lim f) \right)$$

<u>Theorem 26.2</u> Suppose $\{f_n : (a,b) \to \mathbb{R}\}_{n\in\mathbb{N}}$ is a sequence of C^1 functions and suppose $\{f_n'\}_{n\in\mathbb{N}}$ converges uniformly to some function g. Assume further $\exists\, c \in (a,b)$ s.t $\{f_n(c)\}$ converges. Then $\{f_n\}$ converges pointwise to a differentiable function f and $f' = g$. (i.e. $\lim_{n\to\infty} (f_n') = (\lim f_n)'$)

<u>Proof</u> By the fundamental theorem of calculus

$$f_n(x) - f_n(c) = \int_c^x f_n'(t)\,dt$$

By 26.1

$$\int_c^x f_n'(t)\,dt \to \int_c^x g(t)\,dt.$$

Since $f_n(x) = f_n(c) + \int_c^x f_n'(t)\,dt$

$f(x) := \lim_{n\to\infty} f_n(x)$ exists and equals $\lim_{n\to\infty} f_n(c) + \int_c^x g(t)\,dt$.

Then

$$f(c) = \lim_{n\to\infty} f_n(c) + \underbrace{\int_c^c g(t)\,dt}_{=0}$$

and

$$f(x) - f(c) = \int_c^x g(t)\,dt.$$

F.T.C v.2 $\Rightarrow$

$$f'(x) = \frac{d}{dx}\left(\int_c^x g(t)\,dt \right) = g(x) \left(= \lim_{n\to\infty} f_n'(x) \right).$$

$\square$

Thm 26.3 Suppose $a<b$, $c<d$, f is continuous on
$S = [a,b]\times(c,d) \equiv \{(x,y)\in\mathbb{R}^2 \mid a\le x\le b,\ c<y<d\}$.

Assume: $\forall x$, $y \mapsto f(x,y)$ is differentiable and the function
$(x,y) \mapsto \frac{\partial f}{\partial y}(x,y)$ is continuous on S

Then

$$F : (c,d) \to \mathbb{R} \qquad F(y) = \int_a^b f(x,y)\,dx \quad \text{is differentiable}$$

and $$F'(y) = \int_a^b \frac{\partial f}{\partial y}(x,y)\,dx$$

$$\left(\text{ie} \quad \frac{d}{dy}\left(\int_a^b f(x,y)\,dy\right) = \int_a^b \frac{\partial f}{\partial y}(x,y)\,dx \right).$$

Proof Note first that the definition of $F(y)$ makes sense:
Since f is continuous, $\forall y$ $x \mapsto f(x,y)$ is integrable.
To prove that F is differentiable at $y\in(c,d)$ we want/need to show: $\forall \varepsilon>0$ $\exists \delta>0$ so that if $0<|h|<\delta$ then

$$\left| \frac{1}{h}\big(F(y+h)-F(y)\big) - \int_a^b \frac{\partial f}{\partial y}(x,y)\,dx \right| < \varepsilon$$

$$\frac{1}{h}\big(F(y+h)-F(y)\big) = \int_a^b \frac{1}{h}\big(f(x,y+h)-f(x,y)\big)\,dx$$

M.V.T. $\Rightarrow$ $\forall x$ $\exists y_{h,x}$ between $y+h$ and y s.t.

$$f(x,y+h) - f(x,y) = ((y+h)-y)\cdot \frac{\partial f}{\partial x}(x, y_{h,x})$$

ie $$\frac{1}{h}\big(F(y+h)-F(y)\big) = \int_a^b \frac{\partial f}{\partial x}(x, y_{h,x})\,dx$$

Pick $h_0>0$, suff small, s.t. $[y-h_0, y+h_0] \subset (c,d)$.
Since $\frac{\partial f}{\partial y}(x,y)$ is uniformly continuous on $[a,b]\times[y-h_0, y+h_0]$
Therefore, given $\varepsilon>0$ $\exists\delta$ st

$$\left|\frac{\partial f}{\partial y}(x, y_{h,x}) - \frac{\partial f}{\partial y}(x,y)\right| < \varepsilon/(b-a) \quad \text{if } |h|<\delta$$

And then

$$\left| \frac{1}{h}\big(F(y+h)-F(y)\big) - \int_a^b \frac{\partial f}{\partial y}(x,y)\,dx \right|$$

$$= \left| \int_a^b \left(\frac{\partial f}{\partial y}(x,y_{h,x}) - \frac{\partial f}{\partial y}(x,y) \right) dx \right| \le \int_a^b \left| \frac{\partial f}{\partial y}(x,y_{h,x}) - \frac{\partial f}{\partial y}(x,y)\right| dx$$

$$< (b-a)\cdot \frac{\varepsilon}{b-a} = \varepsilon.$$