MATH 424 Eugene Lerman 03/29/2024

27.1

Last time 26.1 $\{f_n : [a,b] \to \mathbb{R}\}$ sequence of integrable function that converge uniformly to $f : [a,b] \to \mathbb{R}$. Then f is integrable and

$$\lim_{n\to\infty} \int_{[a,b]} f_n = \int_{[a,b]} f \quad \left(= \int_{[a,b]} \lim_{n\to\infty} f_n\right)$$

26.2 $\{f_n : [a,b) \to \mathbb{R}\}$ sequence of C^1 functions, $f_n' \to g$ uniformly and $\exists c$ s.t. $\{f_n(c)\}$ converges. Then $\forall x \in (a,b)$ $f(x) := \lim_{n\to\infty} f_n(x)$ exists, f is differentiable and $f' = g$.

Today: a version of Leibniz rule

Theorem 27.1 $R = [a,b] \times [c,d]$, U open set in $\mathbb{R}^2$ containg R. $f : U \to \mathbb{R}$ continuous, $\frac{\partial f}{\partial x} : U \to \mathbb{R}$ exists and is continuous.

Then $F(x) := \int_c^d f(x,y)\, dy$ is differentiable on (a,b) and

$$F'(x) \equiv \frac{d}{dx}\left(\int_c^d f(x,y)\, dy\right) = \int_c^d \frac{\partial f}{\partial x}(x,y)\, dy.$$

Proof Fix $x \in (a,b)$. We want to show

$$\lim_{h\to 0} \frac{1}{h}\left(F(x+h) - F(x)\right) = \int_c^d \frac{\partial f}{\partial x}(x,y)\, dy$$

Since $\frac{\partial f}{\partial x}$ is continuous on R and R is compact, $\frac{\partial f}{\partial x}$ is uniformly continuous. $\Rightarrow$ $\forall \varepsilon > 0$ $\exists \delta > 0$ so that

$$d_2((x,y),(x',y')) < \delta \Rightarrow \left| \frac{\partial f}{\partial x}(x,y) - \frac{\partial f}{\partial x}(x',y') \right| < \varepsilon / d-c$$

By Mean Value Theorem, $\forall (x,y) \in U$ $\forall h$ sufficiently small $\exists \xi = \xi(x,y,h)$ between x and $x+h$ so that

$$\frac{f(x+h,y) - f(x,y)}{(x+h) - x} = \frac{\partial f}{\partial x}(\xi, y)$$

Therefore if $|h| < \delta$ $\quad d_2((\xi,y),(x,y)) \le |h| < \delta$

and then

$$\left| \frac{1}{h} F(x+h,y) - F(x) - \int_c^d \frac{\partial f}{\partial x}(x,y)\,dy \right| =$$

$$= \left| \frac{1}{h}\left(\int_c^d f(x+h,y)\,dy - \int_c^d f(x,y)\,dy\right) - \int_c^d \frac{\partial f}{\partial x}(x,y)\,dy \right|$$

$$= \left| \int_c^d \left(\frac{f(x+h,y)-f(x,y)}{h} - \frac{\partial f}{\partial x}(x,y)\right) dy \right|$$

$$\leq \int_c^d \left| \frac{\partial f}{\partial x}(\xi,y) - \frac{\partial f}{\partial x}(x,y) \right| dy \leq \int_c^d \frac{\varepsilon}{d-c}\,dy = \varepsilon$$

$\square$

<u>Example</u> Compute $\int_0^{\pi} e^{\cos(x)} \cos(\sin x)\,dx$

<u>Solution</u>

We'll use : $\forall z \in \mathbb{C}$ $e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}$ exists and $\frac{d}{dz}(e^z) := \lim_{h\to 0} \frac{e^{z+h}-e^z}{h}$ also exists and equals e^z.

Consequently for any differentiable function $f:(a,b)\to\mathbb{C}$

$$\frac{d}{dx} e^{f(x)} = e^{f(x)} \cdot f'(x).$$ This can also be proved using

$$e^{f(x)} = e^{\mathrm{Re}\, f(x)} \cdot e^{i\,\mathrm{Im}(f(x))}$$

Now let $I(b) = \int_0^{\pi} e^{b\cos x} \cos(b \sin x)\,dx$

Since $\cos(u)$ is even, $e^{b\cos x}\cos(b\sin x)$ is an even function of x

$$\Rightarrow I(b) = \frac{1}{2}\int_{-\pi}^{\pi} b\cos x \cos(b\sin x)\,dx = \frac{1}{2}\int_0^{2\pi} b\cos x \cos(b\sin x)\,dx$$

$$e^{be^{ix}} = e^{b(\cos x + i\sin x)} = e^{b\cos x} \cdot e^{ib\sin x} = e^{b\cos x}\cdot(\cos(b\sin x) + i\sin(b\sin x))$$

$$\Rightarrow b\cos x \cdot \cos(b(\sin x)) = \mathrm{Re}\left(e^{be^{ix}}\right)$$

$$\Rightarrow I(b) = \mathrm{Re}\left(\frac{1}{2}\int_0^{2\pi} e^{be^{ix}}\,dx\right)$$

$$\frac{d}{db}\int_0^{2\pi} e^{b(e^{ix})}\,dx = \int_0^{2\pi} \frac{\partial}{\partial b}\left(e^{b(e^{ix})}\right)dx$$

$$= \int_0^{2\pi} e^{be^{ix}} \cdot \frac{\partial}{\partial b}(be^{ix})\,dx = \int_0^{2\pi} e^{be^{ix}} \cdot e^{ix}\,dx$$

Note $\frac{\partial}{\partial x}\left(e^{be^{ix}}\right) = e^{be^{ix}} \cdot \frac{\partial}{\partial x}(be^{ix}) = e^{be^{ix}} \cdot ib \cdot e^{ix}$

$\Rightarrow$

$$\frac{d}{db}\left(\int_0^{2\pi} \frac{1}{2} e^{be^{ix}} dx\right) = \frac{1}{2i}\int_0^{2\pi} \frac{d}{dx}\left(e^{be^{ix}}\right)dx = \frac{1}{2i}\left(e^{be^{ix}}\right)\Big|_0^{2\pi}$$

$$= \frac{1}{2i}\left(e^{be^{2\pi i}} - e^{be^{0}}\right) = \frac{1}{2}\left(e^{b} - e^{b}\right) = 0$$

$$\Rightarrow \frac{d}{db}[I] = 0$$

$$\Rightarrow \int_0^{\pi} e^{\cos(x)} \cos(\sin x)\, dx = I(1) = I(0) = \int_0^{\pi} e^{0\cos x} \cos(0 \sin x)\, dx$$

$$= \int_0^{\pi} dx = \pi.$$

Aside: Improper integrals.

Suppose $f: \mathbb{R} \to \mathbb{R}$ is integrable on every interval $[a,b]$.

Then we define

$$\int_a^{\infty} f := \lim_{b \to \infty} \int_a^b f \quad \text{if the limit exists}$$

Similarly

$$\int_{-\infty}^{b} f = \lim_{a \to -\infty} \int_a^b f \quad \text{if the limit exists}$$

and

$$\int_{-\infty}^{\infty} f = \lim_{\substack{a \to -\infty \\ b \to +\infty}} \int_a^b f \quad \text{if the limit exists}$$

WARNING

$$\lim_{a \to +\infty} \int_{-a}^{a} \sin(x)\,dx = \lim_{a \to +\infty} 0 \quad \text{but}$$

$$\int_{-\infty}^{\infty} \sin(x)\,dx \text{ does not exist.}$$

Note: if $f: \mathbb{R} \to \mathbb{R}$ is integrable on $[a,b]$ and $f(x) = 0$ for $x \notin [a,b]$, then $\int_{-\infty}^{\infty} f = \int_{[a,b]} f$.

Series

"Recall" Let $\{a_k\}_{k\in\mathbb{N}_0}$ be a sequence of numbers (real or complex)

We define

$$\sum_{n=0}^{\infty} a_n := \lim_{N\to\infty} \sum_{n=0}^{N} a_n \quad \text{provided the limit exists.}$$

If the limit does exist we say

"the series $\sum_{n=0}^{\infty} a_n$ converges"

Note A series need not begin with 0. It can begin with any integer k_0. And then

$$\sum_{k=k_0}^{\infty} a_k = \lim_{N\to\infty} \sum_{k=k_0}^{N} a_k$$

Ex $\sum_{k=0}^{\infty} q^k = 1+q+q^2+\cdots$ Since $1+q+\cdots+q^n = \frac{1-q^{n+1}}{1-q}$ (for $q\neq 1$)

$$\sum_{k=0}^{\infty} q^n = \lim_{N\to\infty} \frac{1-q^{N+1}}{1-q} = \begin{cases} \frac{1}{1-q} & \text{if } |q|<1 \\ \text{does not exist} & \text{otherwise} \end{cases}$$

Definition A series $\sum_{k=0}^{\infty} a_n$ converges absolutely if $\sum_{k=0}^{\infty} |a_k|$ converges (ie, if $\lim_{N\to\infty} \sum_{k=0}^{N} |a_k|$ exists).

Cauchy criterion Since $\mathbb{R}$ and $\mathbb{C}$ are complete

$\sum_{n=0}^{\infty} a_n$ converges $\iff$ $s_n = \sum_{k=1}^{n} a_k$ is Cauchy $\iff$

$\forall \varepsilon>0 \ \exists N$ s.t $n > m-1 > N \implies$

$$\varepsilon > |s_n - s_{m-1}| = \left|\sum_{k=0}^{n} a_n - \sum_{k=0}^{m-1} a_n\right| = \left|\sum_{k=m}^{n} a_n\right|$$

Corollary if $\sum_{n=0}^{\infty} a_n$ converges then $\lim_{n\to\infty} a_n = 0$

Proof By Cauchy's criterion $\forall\varepsilon \ \exists N$ s.t $n > n-1 > N$

$\Rightarrow \varepsilon > |s_n - s_{n-1}| = |a_n|$

ie $\lim_{n\to\infty} a_n = 0$