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28.1

Recall
- $\sum_{n=0}^{\infty} a_n$ converges $\Leftrightarrow$ $\lim_{N\to\infty} \sum_{n=0}^{N} a_n$ exists.
- $\sum_{n=0}^{\infty} a_n$ converges absolutely $\Leftrightarrow$ $\sum_{n=0}^{\infty} |a_n|$ converges.

Cauchy's criterion $\sum_{n=0}^{\infty} a_n$ converges $\Leftrightarrow$ $\forall \varepsilon\ \exists N$ so that if $n > m-1 > N$ then $\left|\sum_{k=m}^{n} a_k\right| < \varepsilon$.

Lemma 28.1 If $\sum_{n=0}^{\infty} a_n$ converges absolutely then $\sum_{n=0}^{\infty} a_n$ converges.

Proof Enough to show: The sequence $s_n := \sum_{k=0}^{n} a_n$ is Cauchy.
Since $\sum_{n=0}^{\infty} a_n$ converges absolutely $\forall \varepsilon > 0\ \exists N$ s.t. for $n > m-1 > N$
$\varepsilon > \left| \sum_{k=m}^{n} |a_k| \right| \left(= \sum_{k=m}^{n} |a_k| \right)$

But $\sum_{k=m}^{n} |a_k| \geq \left|\sum_{k=m}^{n} a_k\right| = |s_n - s_{m-1}|$. $\Rightarrow \{s_n\}$ is Cauchy and we're done. $\square$

Definition A series $\sum_{n=0}^{\infty} a_n$ converges conditionally if it converges but not absolutely, i.e. $\sum_{n=0}^{\infty} |a_n|$ diverges.

Definition A series $\sum b_n$ is a rearrangement of a series $\sum a_n$ if $\exists$ a bijection $f : \mathbb{N} \to \mathbb{N}$ s.t. $b_n = a_{f(n)}$ $\forall n$.

Theorem 28.2 Let $\sum_{n=0}^{\infty} a_n$ be an absolutely convergent series.
Then $\forall$ bijection $f : \mathbb{N} \to \mathbb{N}$ $\sum_{n=0}^{\infty} b_n$ is absolutely convergent where $b_n = a_{f(n)}$
Moreover $\sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} a_n$

Proof Let $a := \sum_{n=0}^{\infty} a_n$, $s_n = \sum_{k=0}^{N} a_k$, $t_n = \sum_{k=0}^{N} b_k$.

Given $\varepsilon > 0$ $\exists N_1$ so that $\varepsilon > \underbrace{\sum_{n=0}^{\infty} |a_n|}_{=a} - \sum_{n=0}^{N_1 - 1} |a_n| = \sum_{n=N_1}^{\infty} |a_n|$
(since $\sum_{n=0}^{k} |a_n| \to \sum_{n=0}^{\infty} |a_n|$)

Then for $N > N_1$ $\quad |a - s_N| = \left|\lim_{M\to\infty} \sum_{n=N+1}^{M} a_n\right| = \lim_{M\to\infty} \left|\sum_{n=N+1}^{M} a_n\right|$

$$\le \lim_{M\to\infty} \sum_{n=N+1}^{M} |a_n| = \sum_{n=N+1}^{\infty} |a_n| < \varepsilon$$

Now choose N_2 so that $\{0, \dots N_1\} \subseteq \{f(0), \dots, f(N_2)\}$

Note that $N_2 \ge N_1$. Now if $N \ge N_2 \ge N_1$ we have

$$|t_N - a| \le |t_N - s_{N_1}| + |s_{N_1} - a| \le |b_0 + \dots + b_N - (a_0 + a_1 + \dots + a_{N_1})| + \varepsilon$$

$$= |a_{f(0)} + a_{f(1)} + \dots + a_{f(N)} - a_0 - a_1 - \dots - a_{N_1}| + \varepsilon$$

each of these cancel with one of the $a_{f(i)}$'s

$$< \sum_{N_1 < n < N} |a_n| + \varepsilon \le \left(\sum_{n=N_1} |a_n|\right) + \varepsilon < 2\varepsilon.$$

Therefore $t_N \to a$.

By the same argument with a_n's replaced with $|a_n|$'s, b_n's with $|b_n|$'s $\sum_{n=0}^{\infty} |b_n|$ exists and equals $\sum_{n=0}^{\infty} |a_n|$.

WARNING

If the series is not absolutely convergent, but convergent, the result is false.

Tests for convergence.

Comparison test (1) Suppose $\sum_{n=0}^{\infty} b_n$ converges, $b_n \ge 0\ \forall n$, $\{a_n\}$ a sequence, and $|a_n| \le b_n\ \forall n$. Then $\sum a_n$ converges absolutely.

(2) Suppose $\{a_n\}$, $\{b_n\}$ are sequences with $0 < a_n \le b_n\ \forall n$ and $\sum a_n$ diverges. Then $\sum b_n$ diverges.

Proof (1) For any n, m with $n \ge m$

$$\sum_{k=m}^{n} |a_k| \le \sum_{k=m}^{n} b_k.$$

Since $\sum b_n$ converges $\forall \varepsilon>0\ \exists N$ s.t. $n>m>N \Rightarrow$

$\varepsilon > \left|\sum_{k=m}^{n} b_k\right| = \sum_{k=m}^{n} b_k$

$\therefore \sum_{k=0}^{\infty} |a_k|$ converges.

(2) $\sum_{k=0}^{n} b_k \ge \sum_{k=0}^{n} a_n \longrightarrow +\infty$.

Root test. Let $\{a_n\}$ be a sequence $\alpha = \lim_{n\to\infty} \sup |a_n|^{1/n} \left(= \lim_{n\to\infty} \sup_{k\ge n} \{ |a_k|^{1/k}\right.$

(i) If $\alpha<1$ then $\sum a_n$ converges absolutely.

(ii) If $\alpha>1$ then $\sum a_n$ diverges

(iii) If $\alpha=1$ the test gives no information.

Proof (i) Suppose $\alpha<1$. Choose $\varepsilon>0$ so that $\alpha+\varepsilon<1$

Since $\lim_{n\to\infty}\left(\sup_{k\ge n} |a_k|^{1/k}\right) = \alpha$ $\exists N$ s.t.

$$\alpha-\varepsilon < \sup_{k>N} |a_k|^{1/k} < \alpha+\varepsilon$$

Hence for $k>N$

$$|a_k|^{1/k} < \alpha+\varepsilon \Rightarrow |a_k| < (\alpha+\varepsilon)^k$$

Since $0<\alpha+\varepsilon$, $\sum_{k=N+1}^{\infty} (\alpha+\varepsilon)^k$ converges.

Comparison test $\Rightarrow \sum_{k=N+1}^{\infty} |a_k|$ converges

$\Rightarrow \sum_{k=0}^{\infty} |a_k|$ converges.

(ii) Since $\lim\sup |a_k|^{1/k} = \alpha$, $\exists$ a subsequence of $|a_n|^{1/n}$ that converges to α. If $\alpha>1$ there is a subsequence a_{n_k} of a_n s.t. $|a_{n_k}|^{1/k} > 1$ eventually $\Rightarrow$

$\exists$ infinitely many indicies n_k s.t. $|a_{n_k}| > 1$

$\Rightarrow a_n \not\to 0. \Rightarrow \sum a_n$ does not converge.

(iii) There are various ways to show that $\lim_{n\to\infty} n^{1/n} = 1$

Consequently $\limsup_{n\to\infty}\left(\frac{1}{n}\right)^{1/n} = \lim_{n\to\infty}\left(\frac{1}{n}\right)^{1/n} = \frac{1}{\lim_{n\to\infty}(n^{1/n})} = 1$

and similarly $\limsup_{n\to\infty}\left(\frac{1}{n^2}\right)^{1/n} = \lim\left(\left(\frac{1}{n}\right)^{1/n}\right)^2 = 1^2.$

But $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges and $\sum_{n=1}^{\infty}\frac{1}{n}$ does not. ∎

Proof that $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges: $\forall n$

$$\sum_{k=1}^{2n}\frac{1}{k} - \sum_{k=1}^{n}\frac{1}{k} = \sum_{k=n+1}^{2n}\frac{1}{k} = \frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+n} \geq \frac{n}{2n} = 1/2$$

$\therefore$ $s_n = \sum_{k=1}^{n}\frac{1}{k}$ is not Cauchy

$\sum\frac{1}{n^2}$ converges by the integral test; see home work

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Ratio test Suppose $\{x_n\}$ is a sequence of real numbers with $x_n \neq 0\ \forall n$

(a) Suppose $\exists r$ with $0<r<1$ and $K\in\mathbb{N}$ s.t.

$$\left|\frac{x_{n+1}}{x_n}\right| \leq r \quad \text{for } n\geq K$$

Then $\sum x_n$ converges absolutely.

(b) Suppose $\exists K$ s.t. $\left|\frac{x_{n+1}}{x_n}\right| \geq 1$ for $n \geq K$.

Then $\sum x_n$ diverges.

Proof (a) By induction $|x_{K+m}| \leq |x_K|\cdot r^m$ for all $m\geq 0$

Since $0<r<1$; $\sum_{m=0}^{\infty}|x_K|r^m$ converges. $\Rightarrow \sum_{m=0}^{\infty}|x_{K+m}|$ converges

by Comparison test $\Rightarrow \sum_{n=0}^{\infty}|x_n|$ converges.

(b) By induction $|x_{K+m}| \geq |x_K| > 0 \quad \forall m.$

$\Rightarrow$ $\{x_n\}$ cannot converge to 0.

$\Rightarrow$ $\sum_{n=0}^{\infty} x_n$ diverges.