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29.1

Last time: - rearrangement of absolutely convergent series

- comparison test

- root test: $\{a_n\}$ (bounded) sequence $\alpha = \limsup_{n\to\infty} |a_n|^{1/n}$

(i) if $\alpha < 1$, $\sum a_n$ converges absolutely

(ii) if $\alpha > 1$ $\sum a_n$ diverges

(iii) if $\alpha = 1$ no information.

Remark about (iii) $\sum \frac{1}{n}$ diverges (we'll prove this soon), $\sum \frac{1}{n^2}$ converges (say by the integral test)

But $\lim_{n\to\infty} \left(\frac{1}{n}\right)^{1/n} = \lim_{n\to\infty} \exp\left(\frac{1}{n} \ln\left(\frac{1}{n}\right)\right) = \exp(0) = 1$

and $\lim_{n\to\infty} \left(\frac{1}{n^2}\right)^{1/n} = \lim_{n\to\infty} \left(\left(\frac{1}{n}\right)^{\frac{1}{n}}\right)^2 = 1$.

Proof that $\sum \frac{1}{n}$ diverges: $\forall n$

$$\sum_{k=1}^{2n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k} = \sum_{k=n+1}^{2n} \frac{1}{k} = \underbrace{\frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{n+n}}_{n} \geq \frac{n}{2n} = \frac{1}{2}$$

$\Rightarrow$ $S_n = \sum_{k=1}^{n} \frac{1}{k}$ is not Cauchy.

Ratio test Suppose $\{x_n\}$ is a sequence of nonzero real numbers.

(i) if $\exists r$ with $0 < r < 1$ and $K \in \mathbb{N}$ so that

$$\left|\frac{x_{n+1}}{x_n}\right| < r \quad \text{for } n \geq K$$

then $\sum x_n$ converges absolutely.

(ii) Suppose $\exists K$ so that $\left|\frac{x_{n+1}}{x_n}\right| \geq 1$ for $n \geq K$

Then $\sum x_n$ diverges

Proof (i) By induction $|x_{K+m}| \leq |x_K| \cdot r^m$ for all $m \geq 0$

Since $0 < r < 1$, $\sum_{m=0}^{\infty} |x_K| r^m$ converges. $\rightarrow$ $\sum_{m=0}^{\infty} |x_{K+m}|$ converges (by Comparison test).

$\Rightarrow$ $\sum_{n=0}^{\infty} |x_n|$ converges

(ii) By induction, $|x_{k+m}| \geq |x_k| > 0 \quad \forall m.$

Hence $\{x_n\}$ cannot converge to 0. $\Rightarrow \sum x_n$ diverges.

<u>Dirichlet test</u> Suppose $\{a_n\}, \{b_n\}$ are sequences, $\{\sum_{n=1}^{N} a_n\}_{N=1}^{\infty}$ bounded

$b_1 \geq b_2 \geq \dots \geq b_n \geq \dots > 0$ and $\lim_{n\to\infty} b_n = 0$.

Then $\sum_{n=1}^{\infty} a_n b_n$ converges.

<u>Example</u> $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges: $a_n = (-1)^n \quad \sum_{n=1}^{N} a_n = \begin{cases} -1 & N \text{ odd} \\ 0 & N \text{ even} \end{cases}$

$\rightarrow S_N = \sum_{n=}^{N} (-1)^n$ is bounded.

$b_n = 1/n$ is decreasing, nonnegative.

Dirichlet test applies.

<u>Note</u> $\sum_{k=1}^{\infty} \frac{(-1)^n}{n}$ is an example of a conditionally convergent series.

~~To~~ <u>Proof of Dirichlet test</u>:

Note that for any two sequences $\{f_k\}, \{g_k\}$ $\forall n > m$

$$\sum_{k=m}^{n} f_k(g_{k+1} - g_k) + \sum_{k=m}^{n} g_{k+1}(f_{k+1} - f_k) = \sum_{k=m}^{n} f_k g_{k+1} - \sum_{k=m}^{n} f_k g_k +$$

$$+ \sum_{k=m}^{n} g_{k+1} f_{k+1} - \sum_{k=m}^{n} g_{k+1} f_k = f_{n+1} g_{n+1} - f_m g_m.$$

$$\Rightarrow \sum_{k=m}^{n} f_k(g_{k+1} - g_k) = f_{n+1} g_{n+1} - f_m \cdot g_m + \sum_{k=m}^{n} g_{k+1}(f_{k+1} - f_k)$$

Now apply this to $f_k = b_k$, $g_k = \sum_{j=1}^{k-1} a_j$. Let $M :=$ upper bound for $\{|g_k|\}_{k=}^{\infty}$

We get

$$\left|\sum_{k=m}^{n} a_k b_k\right| = \left|\sum_{k=m}^{n} b_k (g_{k+1} - g_k)\right| = \left| b_{n+1}\left(\sum_{j=1}^{n} a_j\right) - b_m \left(\sum_{j=1}^{m-1} a_j\right)\right.$$

$$\left. - \sum_{k=m}^{n} g_{k+1}(b_{k+1} - b_k)\right| \leq M|b_{n+1}| + M|-b_m| + M \sum_{k=m}^{n} (b_k - b_{k+1})$$

$$= M(b_{n+1} + b_m + b_m - b_{n+1}) = 2 b_m \cdot M \xrightarrow[\text{as } m \to \infty]{} 0$$

$\Rightarrow \sum a_k b_k$ converges by Cauchy's criterion.

<u>Definition</u> Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of real numbers, $x_0 \in \mathbb{R}$. The series $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ is called a <u>power series</u> centered at x_0.

$f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$ is a function with domain $\{x \mid \sum_{n=0}^{\infty} a_n (x-x_0)^n$ converges$\}$.

<u>Ex</u> $\sum_{n=0}^{\infty} (x-3)^n = \frac{1}{1-(x-3)}$ on $\{x \mid |x-3| < 1\} = (2,4)$

<u>Ex</u> $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all $x \in \mathbb{R}$ by <u>(Corollary to) Taylor's thm.</u>

<u>Recall</u> 19.1 : $f \in C^{\infty}(-a,a)$, $\exists M, C$ s.t. $|f^{(k)}(x)| < MC^k$

Then $f(x) = \sum \frac{f^{(k)}(0)}{k!} x^k$ for all $x \in (-a,a)$.

Now,

$(e^x)' = e^x \Rightarrow \frac{d^k}{dx^k} e^x = e^x \ \forall k$. Also, $\forall a$, $\forall x \in (-a,a)$

$|e^x| \le e^a =: M$

$$\Rightarrow e^x = \sum_{k=0}^{\infty} \frac{e^0}{k!} x^k \text{ on } (-a,a) \text{ for any } a.$$

ie.

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k \quad \forall x \in \mathbb{R}$$

<u>Theorem 29.1</u> Given a power series $\sum a_n (x-x_0)^n$.

$\beta = \limsup |a_n|^{1/n}$ and $R = 1/\beta$.

(If $\beta = 0$ set $R = +\infty$. If $\beta = +\infty$, set $R = 0$)

(i) the power series $\sum a_n (x-x_0)^n$ converges for all x with $|x-x_0| < R$

(ii) The power series diverges if $|x-x_0| > R$

R is called the radius of convergence of the power series.

<u>Proof</u> We apply root test: $\limsup |a_n (x-x_0)^n|^{1/n} = \limsup |a_n|^{1/n} |x-x_0|$

$= |x-x_0| \limsup_{n\to\infty} |a_n|^{1/n} = |x-x_0|\beta$

Therefore $\limsup |a_n(x-x_0)^n|^{1/n} < 1$ iff $|x-x_0|\beta < 1$ iff $|x-x_0| < R$.

and the series converges absolutely

Similarly $|x-x_0| > R \iff \limsup |a_n(x-x_0)^n|^{1/n} > 1$

and the series diverges

Ex $\sum_{n=1}^{\infty} \frac{x^n}{n}$ $\limsup |a_n|^{1/n} = \limsup n^{-1/n} = 1. \Rightarrow R = 1/1 = 1$

The series converges absolutely on $(-1,1)$, diverges if $|x|>1$

if $x=1$ $\sum \frac{x^n}{n} = \sum \frac{1}{n}$ diverges

if $x=-1$, $\sum \frac{(-1)^n}{n}$ converges conditionally.

Ex $\sum_{k=0}^{\infty} 3^{-k}(x-5)^{2k}$ $a_k = \begin{cases} 3^{-k/2} & k \text{ even} \\ 0 & k \text{ odd} \end{cases}$

$\limsup |a_k|^{1/k} = 3^{-1/2}$. $\Rightarrow R = \sqrt{3}$

$\Rightarrow$ The series converges absolutely on $(5-\sqrt{3}, 5+\sqrt{3})$

diverges if $|x-5| \geq \sqrt{3}$

Corollary 29.2 If $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges at $y \in \mathbb{R}$

then the radius of convergence R satisfies $R \geq |y-x_0|$

In particular the series converges absolutely on $(x_0 - |y-x_0|, x_0 + |y-x_0|)$

Proof If $|z-x_0| > R$ then the series diverges at z.

Since it converges at y_0, we must have $|y-x_0| \leq R$.