Definition A <u>metric space</u> is a set E together with a function $d: E \times E \to [0,\infty)$ so that $\forall x, y, z \in E$

1) $d(x,y) = 0 \iff x = y$ (nondegeneracy)
2) $d(x,y) = d(y,x)$ (symmetry)
3) $d(x,z) \le d(x,y) + d(y,z)$ (triangle inequality)

Thus a metric space is a pair (E, d).

- The function d is called a metric (think distance).

Examples (i) $E = \mathbb{Q}$ $\quad d(x,y) = |x-y|$

(ii) $E = \mathbb{R}$ $\quad d(x,y) = |x-y|$

(iii) Let E be any set. Define $d: E \times E \to [0,\infty)$ by
$$d(x,y) = \begin{cases} 1 & x \ne y \\ 0 & x = y. \end{cases}$$

(iv) $E = \mathbb{C}$. $d(x+iy, u+iv) = |(x+iy) - (u+iv)| = \sqrt{(x-u)^2 + (y-v)^2}$.

Since $\mathbb{C} = \mathbb{R}^2$, (iv) is a special case of

(v) $E = \mathbb{R}^n$, $\quad d_2(x,y) := \left(\sum_{i=1}^{n} (x_i - y_i)^2\right)^{1/2}$, Euclidean distance

It's not obvious that d_2 satisfies the triangle inequality.

(vi) $E = \mathbb{R}^n$ $\quad d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i|$

$$d_\infty(x,y) = \max\{|x_1-y_1|, \dots, |x_n - y_n|\}$$

Remark If (E,d) is a metric space and $\tilde{E} \subseteq E$ then $(\tilde{E}, d|_{\tilde{E}\times\tilde{E}})$ is also a metric space, a metric <u>subspace</u> of E. That is we measure distance between points in $\tilde{E}$ by measuring the distance in E.

There are many other examples...

Definition The ℓ^2-norm on $\mathbb{R}^n$ is the function $\|\cdot\|_2 : \mathbb{R}^n \to [0,\infty)$
$$\|x\|_2 := \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$$

<u>Note</u> $\quad d_2(x,y) = \|x-y\|_2$

We may write <u>$\|x\|$</u> for $\|x\|_2$.

Theorem (Cauchy-Schwarz inequality) For all $x, y \in \mathbb{R}^n$

(*) $\left| \sum_{i=1}^{n} x_i y_i \right| \leq \|x\| \cdot \|y\|$.

Proof If x or y are zero, we get $0 \leq 0$, so true.

Now fix $x, y \in \mathbb{R}^n$, $x, y \neq 0$.

For any $\alpha \in \mathbb{R}$ $x - \alpha y = (x_1 - \alpha y_1, \dots, x_n - \alpha y_n)$ satisfies

$$0 \leq \|x - \alpha y\|^2 = \sum_{i=1}^{n} (x_i - \alpha y_i)^2 = \sum (x_i^2 - 2\alpha x_i y_i + \alpha^2 y_i^2)$$
$$= \|x\|^2 + \alpha^2 \|y\|^2 - 2\alpha \left(\sum x_i y_i\right)$$

For $\alpha = \pm \frac{\|x\|}{\|y\|}$ we get

$$0 \leq \|x\|^2 + \frac{\|x\|^2}{\cancel{\|y\|^2}} \cancel{\|y\|^2} \mp 2 \frac{\|x\|}{\|y\|} \left(\sum x_i y_i\right)$$

$$\Rightarrow$$

$$\Rightarrow \quad \pm \left(\sum x_i y_i\right) \leq \|x\| \|y\|$$

$$\Rightarrow \quad \left| \sum x_i y_i \right| \leq \|x\| \|y\|.$$

Theorem 3.1. $\forall x, y \in \mathbb{R}^n$ $\|x + y\| \leq \|x\| + \|y\|$

Consequently $\forall x, y, z \in \mathbb{R}^n$

$$d_2(x, z) \leq d_2(x, y) + d_2(y, z)$$

(recall: $d_2(x, y) = \|x - y\|$ etc.)

Proof

(i) $\|x + y\|^2 = \sum (x_i + y_i)^2 = \sum (x_i^2 + 2x_i y_i + y_i^2)$

$$= \left(\sum x_i^2\right) + 2 \sum x_i y_i + \left(\sum y_i\right)^2 \leq \|x\|^2 + 2\left|\sum x_i y_i\right| + \|y_i\|^2$$
$$\leq \|x\|^2 + 2\|x\| \|y\| + \|y\|^2 \quad \text{by Cauchy-Schwarz}$$
$$= (\|x\| + \|y\|)^2.$$

(ii)

$$d_2(x, z) = \|x - z\| = \|(x - y) + (y - z)\| \leq \|x - y\| + \|y - z\|$$
$$= d_2(x, y) + d_2(y, z). \quad \square$$

Corollary 3.2 $(\mathbb{R}^n, d_2)$ is a metric space.

Proof We proved the triangle inequality in 3.1

$$d_2(x, y) = \left(\sum (x_i - y_i)^2\right)^{1/2} = \left(\sum (y_i - x_i)^2\right)^{1/2} = d_2(y, x)$$

$$0 = d_2(x, y) \iff \left(\sum (x_i - y_i)^2\right)^{1/2} = 0 \iff \sum (x_i - y_i)^2 = 0 \iff$$

$(x_i - y_i)^2 = 0$ for each i $\iff$ $x = (x_1 \dots x_n) = (y_1 \dots y_n) = y$. $\square$

From now on we may write $\mathbb{R}^n$ instead of $(\mathbb{R}^n, d_2)$ with d_2 understood

Exercise: Read and understand two propositions on p 37 of the text.

Definition: Let (E, d) be a metric space. An open ball of radius $r > 0$ centered at $x \in E$ is the set

$$B_r(x) \equiv B(x, r) := \{ y \in E \mid d(x,y) < r \}$$

A closed ball of radius r centered at x is

$$\overline{B(x,r)} := \{ y \in E \mid d(x,y) \leq r \}.$$

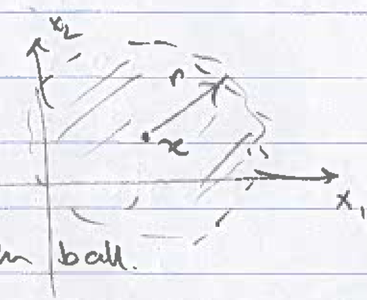
Ex: $E = \mathbb{R}$, $d(x,y) = |x - y|$.

$$B_r(x) = \{ y \in \mathbb{R} \mid |x-y| < r \} = \{ y \in \mathbb{R} \mid -r < y - x < r \}$$

$$= (x - r, x + r)$$

$$\overline{B(x,r)} = [x - r, x + r].$$

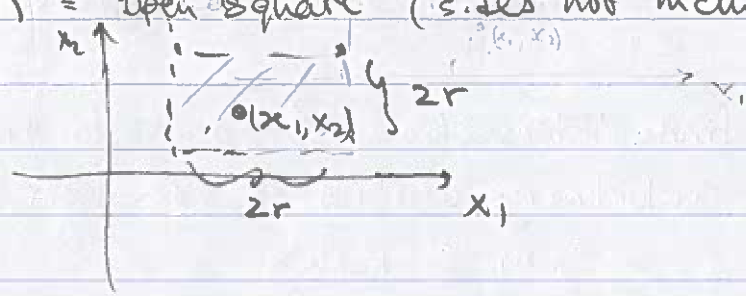
Ex: $E = \mathbb{R}^2$, $d = d_2$, $B(x, r) =$



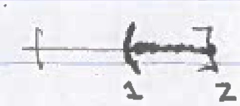
Standard round open ball.

$E = \mathbb{R}^2$, $d(x,y) = d_\infty(x,y) = \max(|x_1 - y_1|, |x_2 - y_2|)$

$B(x,r) =$ open square (sides not included) of the sort



Ex: $E = [0, 2]$, $d(x,y) = |x - y|$, $B(2,1) =$



$= (1, 2]$

Ex E = a set $\quad d(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$

$B_2(x) \equiv B(x,2) = E$

$B_1(x) = \{x\}.$

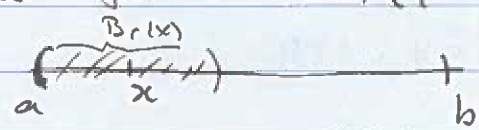
Definition (most important!) A subset U of a metric space (E,d) is open if $\forall x \in U$ $\exists r = r(x)$ so that

$$B_{r(x)}(x) \subseteq U.$$

Ex $E = \mathbb{R}$, $d(x,y) = |x-y|$ $\quad U = (a,b)$ is open because

$\forall x \in (a,b)$ $\quad B_r(x) = (x-r, x+r)$

and for $r = \min(|x-a|, |x-b|)$ $\quad (x-r, x+r) \subseteq (a,b)$.



Definition A subset F of a metric space (E,d) is closed if its complement $F^c \equiv E \setminus F$ is open.

Ex $[0,1] \subseteq \mathbb{R}$ is closed.

$[0,1) \subseteq \mathbb{R}$ is neither open nor closed.

(sets are not doors; they don't need to be either open or closed. They can be neither).

Remark For any metric space (E,d)

$\emptyset$ and E are both open and closed.

Next time - finite intersections of open sets are open

- arbitrary unions of open sets are open.

(see p 39 of book)