MATH 424 Eugene Lerman

Recall Given a power series $\sum a_n (x-x_0)^n$, $\{a_n\}$ a sequence in $\mathbb{R}$, $x_0 \in \mathbb{R}$, we set $\beta = \limsup |a_n|^{1/n}$ and $R := 1/\beta$.

We proved:

i) $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ converges absolutely for all x with $|x-x_0| < R$

ii) $\sum a_n (x-x_0)^n$ diverges if $|x-x_0| > R$

(if $|x-x_0| = R$, it depends)

Lemma 30.1 Let $\{a_n\}$ be a sequence, ($a_n \neq 0\ \forall n$) if $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L$, then $L = \limsup |a_n|^{1/n}$

Proof (In the case where $0 < L < \infty$; other cases are left as exercise)

Since $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L$, $\forall \varepsilon$ with $0 < \varepsilon < L$ $\exists N$ so that if $n \geq N$ then

$$L - \varepsilon < \frac{|a_{n+1}|}{|a_n|} < L + \varepsilon$$

$$\Rightarrow (L-\varepsilon)|a_N| < |a_{N+1}| < (L+\varepsilon)|a_N|$$

$$\left((L-\varepsilon)^2 |a_N| <\right) (L-\varepsilon)|a_{N+1}| < |a_{N+2}| < (L+\varepsilon)|a_{N+1}| \left(< (L+\varepsilon)^2 |a_N|\right)$$

...

$$\Rightarrow \forall k > N \quad |a_N|(L-\varepsilon)^{k-N} < |a_k| < |a_N|(L+\varepsilon)^{k-N}$$

$$\Rightarrow |a_N|^{1/k} (L-\varepsilon)^{1-N/k} < |a_k|^{1/k} < |a_N|^{1/k} (L+\varepsilon)^{1-N/k}$$

$$\Rightarrow \underbrace{\lim_{k\to\infty} |a_N|^{1/k} (L-\varepsilon)^{1-N/k}}_{= 1\cdot(L-\varepsilon)} \leq \limsup |a_k|^{1/k} \leq \underbrace{\lim_{k\to\infty} |a_N|^{1/k} (L+\varepsilon)^{1-N/k}}_{1\cdot(L+\varepsilon)}$$

$$\Rightarrow \limsup_{k\to\infty} |a_k|^{1/k} = L.$$

Example $\sum_{k=1}^{\infty} \frac{(x-2)^k}{k^2}$.

$x_0 = 2$, $a_k = 1/k^2$, $\frac{1}{R} = \limsup \left(\frac{1}{k^2}\right)^{1/k} = \lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|}$

$$\Rightarrow \frac{1}{R} = \lim_{k\to\infty} \frac{\frac{1}{(k+1)^2}}{1/k^2} = \lim_{k\to\infty} \left(\frac{k}{k+1}\right)^2 = 1$$

$\Rightarrow$ The series converges on $(2-1, 2+1) = (1,3)$.

Ex $\sum_{n=0}^{\infty} \frac{x^n}{n!}$. $a_n = \frac{1}{n!}$. $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} = 0$

$\Rightarrow R = 1/0 = +\infty$. The series converges for all x

Ex $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$. Interval of convergence?

Consider $g(y) = \sum_{n=0}^{\infty} \frac{(-1)^n y^n}{(2n)!}$. Then $f(x) = g(y^2)$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1/(2n+2)!}{1/(2n)!} = \frac{\cancel{(2n)!}}{\cancel{2n!}(2n+1)(2n+2)} \xrightarrow[n\to\infty]{} 0$$

$\Rightarrow g(y)$ converges for all y. $\Rightarrow f(x) = g(x^2)$ converges for all x.

< do Remark on p 30.3 first >

Theorem (Weierstrass M-test). Suppose $\{M_k\}_{k=0}^{\infty}$ is a sequence of nonnegative real numbers so that $\sum_{k=0}^{\infty} M_k$ converges. Let $\{g_k : D \to \mathbb{R}\}$ be a sequence of functions so that $\forall k$ $|g_k(x)| \le M_k$ for all $x \in D$. Then

$$\sum_{k=0}^{\infty} g_k(x) \text{ converges uniformly on } D.$$

Proof Since the series $\sum_{k=0}^{\infty} M_k$ converges $\forall \varepsilon > 0$ $\exists N$ so that

$$m \ge n-1 \ge N \Rightarrow \sum_{k=n}^{m} M_k < \varepsilon$$

Hence $\forall x \in D$

$$\left|\sum_{k=n}^{m} g_k(x)\right| \le \sum_{k=n}^{m} |g_k(x)| \le \sum_{k=n}^{m} M_k < \varepsilon.$$

$\square$

Ex Consider the series $\sum_{k=0}^{\infty} 2^{-k} x^{k^2}$ on $[-1,1]$.

$\forall x \in [-1,1]$ $\quad |2^{-k} x^{k^2}| \le 2^{-k}$.

Let $M_k = 2^{-k}$. Since $\sum_{k=0}^{\infty} 2^{-k} = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k$ converges

the series $\sum 2^{-k} x^{k^2}$ converges uniformly on $[-1,1]$

by Weierstrass M-test.

Notation For $A \subseteq \mathbb{R}$, $\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$

χ_A is called the indicator function of A.

Remark

Recall: a sequence $\{f_n : D \to \mathbb{R}\}$ converges uniformly to a function $f : D \to \mathbb{R}$ $\iff$ $\forall \varepsilon > 0$ $\exists N$ so that if $n, m \geq N$ then

$$|f_n(x) - f_m(x)| < \varepsilon \quad \forall x \in D$$

Hence a series $\sum_{k=0}^{\infty} g_n(x)$ converges uniformly $\iff$ $\forall \varepsilon > 0$ $\exists N$ s.t. $m > n-1 \geq N$ we have

$$\left|\sum_{k=n}^{m} g_k(x)\right| < \varepsilon. \qquad \text{Cauchy criterion}$$

(In particular $\forall n > N$, we must have
$$|g_n(x)| < \varepsilon \;)$$

Ex $\quad g_k(x) = x\,\chi_{[\frac{1}{k+1}, \frac{1}{k})}(x) \qquad \forall x \in [0,1)$

$$\text{Then} \quad \sum_{k=1}^{N} g_k(x) = x\left(\chi_{[\frac{1}{2},1)} + \chi_{[1/3,1/2)} + \cdots + \chi_{[\frac{1}{N+1},\frac{1}{N})}\right)$$
$$= x\,\chi_{[\frac{1}{N+1},1)}$$

$$\Rightarrow \sum_{k=1}^{\infty} g_k(x) = \lim_{N\to\infty} \sum_{k=1}^{N} g_k(x) = \lim_{N\to\infty} x\,\chi_{[\frac{1}{N+1},1)} = x \quad \forall x \in (0,1)$$

Claim $\sum_{k=1}^{\infty} g_k(x)$ converges uniformly for all $x \in (0,1)$.

Reason Cauchy criterion. Given $\varepsilon > 0$ choose $N > \frac{1}{\varepsilon}$. Then for $m > n-1 > N$,

$$\left|\sum_{k=n}^{m} g_k(x)\right| = x\,\chi_{[\frac{1}{m+1},\frac{1}{n})} \le \frac{1}{n} < \frac{1}{N} < \varepsilon$$

<u>Note</u> $\sup_{x\in(0,1)} g_k(x) = \frac{1}{k}$ and $\sum_{k=1}^{\infty} \frac{1}{k}$ does <u>not</u> converge.
Consequently
Weierstrass M-test does not apply. None the less the series of functions $\sum g_k(x)$ does converge uniformly <u>on $(0,1)$</u>.

<u>Recall</u> <u>Thm 26.1</u> $\{f_n : [a,b] \to \mathbb{R}\}$ sequence of integrable functions so that $f_n \to f$ uniformly. Then f is integrable and
$$\int_{[a,b]} f = \lim_{n\to\infty} \int_{[a,b]} f_n$$

<u>Corollary 30.2</u> Suppose $\sum_{n=0}^{\infty} f_n(x)$ is a series of integrable functions on $[a,b]$ and the series converges uniformly to f. Then
$$\int_{[a,b]} \left(\sum_{n=0}^{\infty} f_n(x)\right) = \sum_{n=0}^{\infty} \int_{[a,b]} f_n$$

<u>Proof</u> Let $S_n = \sum_{k=0}^{n} f_k(x)$. Then $S_n \to f \equiv \sum_{n=0}^{\infty} f_n(x)$ uniformly.
Hence
$$\int_{[a,b]} f \equiv \int_{[a,b]} \sum_{n=0}^{\infty} f_n(x) = \lim_{n\to\infty} \int_{[a,b]} \sum_{k=0}^{n} f_k(x)$$
$$= \lim_{n\to\infty} \sum_{k=0}^{n} \int_{[a,b]} f_k \equiv \sum_{k=0}^{\infty} \left(\int_{[a,b]} f_k\right).$$
$\square$

I forgot to prove a corollary to Thm 29.1: Suppose $f(x) = \sum a_n (x-x_0)^n$ is a power series with $\frac{1}{R} = \limsup |a_n|^{1/n}$ and $R > 0$.
If $0 < R_1 < R$ then $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ converges <u>uniformly</u> on $[-R_1, R_1]$.

The proof follows easily from Weierstrass M-test.
We'll do it next time.