MATH 424 Eugene Lerman 04/12/2024

Last time Weierstrass M-test: $\{M_k\}_{k=0}^{\infty}$ sequence of non-negative numbers s.t. $\sum_{k=0}^{\infty} M_k$ converges $\{g_k : D \to \mathbb{R}\}$ sequence of functions so that $\forall k\ \forall x \in D$ $|g_k(x)| \le M_k$. Then $\sum_{k=0}^{\infty} g_k(x)$ converges uniformly on D.

Cor 30.2

$\{f_n : [a,b] \to \mathbb{R}\}$ sequence of integrable functions so that $\sum_{k=0}^{\infty} f_k(x)$ converges for all $x \in [a,b]$ and the converges is uniform. Then

$$\int_{[a,b]} \left(\sum_{n=0}^{\infty} f_n(x) \right) = \sum_{n=0}^{\infty} \int_{[a,b]} f_n$$

Recall Thm 29.1 $f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ a power series, $1/R = \limsup |a_n|^{1/n}$, $R > 0$. Then $\sum a_n (x - x_0)^n$ converges for all $x \in (x_0 - R, x_0 + R)$.

Cor 31.1 (to 29.1) $\forall R_1$ with $0 < R_1 < R$, $\sum a_n (x - x_0)^n$ converges uniformly on $[-R_1 + x_0, R_1 + x_0]$.

Proof Consider the power series $\sum_{k=0}^{\infty} |a_k| y^k$. Then

$$\limsup (|a_k|)^{1/k} = \limsup |a_k|^{1/k}$$

$\Rightarrow$ $\sum |a_k| y^k$ and $\sum a_k (x - x_0)^k$ have the same radius of convergence. $\rightarrow$ $\forall R_1$ with $0 < R_1 < R$, $\sum |a_k| R_1^k$ converges. Since $|a_k (x - x_0)^k| \le |a_k| R_1^k$ for all $x \in [-R_1 + x_0, R_1 + x_0]$ $\sum a_k (x - x_0)^k$ converges uniformly on $[x_0 - R_1, x_0 + R_1]$ by Weierstrass M-test. □

Theorem 31.2 Suppose $f(x) = \sum a_n x^n$ has radius of convergence $R > 0$. [i.e. $\frac{1}{R} = \limsup |a_n|^{1/n}$ ...]

Then $\forall x \in (-R, R)$

$$\int_0^x \left(\sum_{n=0}^{\infty} a_n t^n \right) dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^n.$$

<u>Proof</u> $S_n(t) = \sum_{k=0}^{n} a_k t^k \xrightarrow{\text{uniformly}} f(t)$ on $[-|x|, |x|]$ by 31.1.

Each $S_n(t)$ is integrable. Hence $f(t)$ is integrable on $[-|x|, |x|]$ and

$$(*) \quad \int_0^x S_n(t)\,dt \xrightarrow{n\to\infty} \int_0^x f(t)\,dt$$

Since $\int_0^x \left(\sum_{k=0}^{n} a_k t^k\right) dt = \sum_{k=0}^{n} \frac{a_k}{k+1} x^{k+1}$, $(*)$ says:

$$\sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1} = \int_0^x \left(\sum_{k=0}^{\infty} a_k t^k\right) dt \qquad \square$$

What happens to the radius of convergence when we integrate term by term?

<u>Lemma 31.3</u> Suppose the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$ is R. Then the radiuses of convergence of the series $\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$ and of $\sum_{n=0}^{\infty} n a_n x^{n-1}$ are also R.

<u>Proof</u> $\left(\sum_{n=0}^{\infty} n a_n x^{n-1}\right) \cdot x = \sum_{n=0}^{\infty} n a_n x^n$ and

$$\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} = x \cdot \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^n.$$

Now $\limsup |n a_n|^{1/n} = \lim_{n\to\infty} (n^{1/n}) \cdot \limsup |a_n|^{1/n} = 1 \cdot \frac{1}{R}$

and

$$\limsup \left|\frac{a_n}{n+1}\right|^{1/n} = \lim_{n\to\infty} \left(\frac{1}{n+1}\right)^{1/n} \limsup |a_n|^{1/n} = 1 \cdot \frac{1}{R}$$

where we used

<u>Fact</u> If $s_n \to s > 0$ and $\{t_n\}$ is any sequence then

$$\limsup s_n t_n = \left(\lim_{n\to\infty} s_n\right) \cdot \limsup t_n.$$

Proof exercise

<u>Hint</u> $t = \limsup t_n \iff \exists$ a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ so that $t_{n_k} \to t$.

Thm 31.4 Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $R > 0$. Then f is differentiable on $(-R, R)$ and
$$f'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1} \quad \text{for all } x \in (-R, R).$$

Proof By 31.3, $g(x) = \sum n a_n x^{n-1}$ has radius of convergence R. Consider $f_n(x) = \sum_{k=0}^{n} a_k x^k$.

Then $f_n'(x) = \sum_{k=0}^{n} k a_k x^{k-1}$

By 31.1 $\{f_n'(x)\}$ converges uniformly to g on $[-R', R']$ for any R' with $R' < R$.

$\Rightarrow$ $f(x) = \lim_{n\to\infty} f_n(x)$ is differentiable on $(-R', R')$

and $f'(x) = g(x)$ on $(-R', R')$ $\forall R'$.

$\Rightarrow$ $\sum_{n=0}^{\infty} a_n x^n$ is differentiable on $(-R, R)$ and
$$\left(\sum_{n=0}^{\infty} a_n x^n\right)' = \sum_{n=0}^{\infty} n a_n x^{n-1}.$$

Ex $f(t) = \sum_{n=0}^{\infty} (-1)^n t^n$. The radius of convergence of f is 1.

and $f(t) = \sum_{n=0}^{\infty} (-t)^n = \dfrac{1}{1+t}$ $\forall t \in (-1, 1)$.

$$\Rightarrow \ln(1+x) = \int_1^{1+x} \frac{dt}{t} = \int_0^x \frac{dt}{1+t} = \int_0^x \left(\sum_{n=0}^{\infty} (-1)^n t^n\right) dt$$

$$= \sum_{n=0}^{\infty} (-1)^n \left(\int_0^x t^n dt\right) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$$

Ex $f(x) = \sum_{n=0}^{\infty} \dfrac{x^n}{n!}$. Radius of convergence is $+\infty$.

$$\Rightarrow f'(x) = \sum_{n=0}^{\infty} \left(\frac{x^n}{n!}\right)' = \sum_{n=0}^{\infty} \frac{n}{n!} x^{n-1} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Ex <u>Define</u> $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$, $\cos x = \sum \frac{(-1)^n x^{2n}}{(2n)!}$

Radius of convergence of both series is $+\infty$.

$\sin(0) = \sum_{n=0}^{\infty} \frac{(-1)^n 0^{2n+1}}{(2n+1)!} = 0$, $\cos x = \frac{(-1)^0}{0!} +$ higher order terms

hence $\cos(0) = \frac{(-1)^0}{0!} = \frac{1}{1} = 1$.

Also easy to see: $\sin(-x) = -\sin x$, $\cos(-x) = \cos x$.

$\sin(x)$, $\cos(x)$ are C^∞, $(\sin(x))' = \cos x$, $(\cos x)' = -\sin x$.

$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{(-1)^n x^{2n}}{(2n+1)!} = 1$, $\lim_{x\to 0} \frac{1-\cos x}{x} = 0$.

<u>Lemma 31.5</u> $\cos^2 x + \sin^2 x = 1$ for all $x \in \mathbb{R}$

<u>Proof</u> $\cos^2(0) + \sin^2(0) = 1^2 + 0^2 = 1$

$$\frac{d}{dx}(\cos^2 x + \sin^2 x) = 2\cos x \cdot (\cos x)' + 2\sin x \cdot (\sin x)'$$
$$= 2\cos x \cdot (-\sin x) + 2\sin x \cdot \cos x = 0$$

$\therefore$ $\cos^2 x + \sin^2(x) = \cos^2(0) + \sin^2(0) = 1$ for all x.

One then <u>defines</u> π by $\frac{\pi}{2} = \inf\{x \in (0,\infty) \mid \cos x = 0\}$

shows that π exists and $\sin x$, $\cos x$ are periodic functions

of period 2π and so on ...