(I will try to follow Jim Belk's Measure theory course)

<u>Definition</u> Let S be a set, $f: S \to [0,\infty]$ a function. The <u>sum of f over S</u> is

$$\sum_{s\in S} f(s) := \sup\{ f(s_1)+\dots+f(s_n) \mid n>0, \{s_1,\dots,s_n\} \subseteq S \text{ finite subset}\}$$

For any $\{s_1,\dots,s_n\} \subseteq S$, $\sum_{i=1}^{n} f(s_i)$ is a <u>finite partial sum</u> of $\sum_{s\in S} f(s)$

<u>Note</u> f is not allowed to take negative values

Sanity check

<u>Proposition 32.1</u> For any $f: \mathbb{N} \to [0,\infty]$, $\sum_{s\in\mathbb{N}} f(s) = \sum_{n=1}^{\infty} f(n)$.

<u>Proof</u>

$$\sum_{s\in\mathbb{N}} f(s) \geq \sum_{n=1}^{N} f(n) \quad \forall N \in \mathbb{N}. \text{ Hence}$$

$$\sum_{s\in\mathbb{N}} f(s) \geq \lim_{N\to\infty} \sum_{n=1}^{N} f(n) = \sum_{n=1}^{\infty} f(n).$$

On the other hand, given $\{n_1,\dots,n_k\} \subseteq \mathbb{N}$ let $N = \max\{n_1,\dots,n_k\}$.

Then $f(n_1)+\dots+f(n_k) \leq \sum_{k=1}^{N} f(k) \leq \sum_{n=1}^{\infty} f(n)$

Hence $\sum_{s\in\mathbb{N}} f(s) = \sup\{ f(n_1)+\dots+f(n_k) \mid \{n_1,\dots,n_k\} \subseteq \mathbb{N}\} \leq \sum_{n=1}^{\infty} f(n)$

<u>Proposition 32.2</u> Suppose S is uncountable, $f: S \to [0,\infty]$. If

$$\sum_{s\in S} f(s) < \infty$$

then $f(s) = 0$ for all but countably many $s \in S$.

<u>Proof</u> Since $\sum_{s\in S} f(s) < \infty$, $\forall n \in \mathbb{N}$ $S_n := \{ s\in S \mid f(s) \geq \frac{1}{n}\}$ must be <u>finite</u> for otherwise $\sum_{s\in S_n} f(s)$ is infinite

But then $\bigcup_{n\in\mathbb{N}} S_n$ is a countable union of <u>finite sets</u> hence countable.

Since $\bigcup_{n\in\mathbb{N}} S_n = \{ s\in S \mid f(s) > 0\}$

we're done. $\square$

<u>Definition/notation</u> $A = B \sqcup C$ iff $A = B \cup C$ and $B \cap C = \emptyset$

A is a <u>disjoint union</u> of B and C. [Belk uses $\uplus$ for $\sqcup$]

Similarly $A = \coprod_{n \in \mathbb{N}} S_n$ if $A = \bigcup_{n \in \mathbb{N}} S_n$ and $S_i \cap S_j = \emptyset$ for $i \neq j$.

<u>Given</u> $(a,b) \in \mathbb{R}$ we define its <u>length</u> $l((a,b)) := b - a$.

<u>Question</u> Can we extend l to a function $m : \underset{\text{set of subsets of } \mathbb{R}}{P(\mathbb{R})} \to [0,\infty]$

so that

1) $m((a,b)) = l((a,b)) = b - a$. $\forall$ $a < b$.

2) $\forall$ $S = \coprod_{i \in \mathbb{N}} S_i$

$$m\left(\coprod S_i\right) = \sum_{i \in \mathbb{N}} m(S_i) \qquad ?$$

<u>Answer</u> Axiom of choice says "no".

That's a problem. Solution: restrict the domain of m ("measure") to a subcollection $\mathcal{M} \subset P(\mathbb{R})$.

$\mathcal{M}$ = set of Lebesgue measurable sets.

<u>Main Theorem</u> $\exists$ a collection/set $\mathcal{M}$ of subsets of $\mathbb{R}$ [the measurable sets] and a function $m : \mathcal{M} \to [0,\infty]$ so that

1) $m((a,b)) = b - a$

2) $\forall$ $E \in \mathcal{M}$, $E^c \equiv \mathbb{R} \setminus E \in \mathcal{M}$

3) $\forall$ countable collection $(E_n)_{n \in \mathbb{N}}$ of measurable sets $\bigcup_{n \in \mathbb{N}} E_n \in \mathcal{M}$. Moreover if $E_i \cap E_j = \emptyset$ for $i \neq j$, i.e. $\bigcup_{n \in \mathbb{N}} E_n = \coprod_{n \in \mathbb{N}} E_n$ then

$$m\left(\bigcup_{n \in \mathbb{N}} E_n\right) = \sum_{n \in \mathbb{N}} m(E_n)$$

First step in proving main Theorem (existence of Lebesgue measurable sets and of Lebesgue measure)

Lebesgue outer measure $m^*: \mathcal{P}(\mathbb{R}) \to [0,\infty]$

Definition For any set $S \subseteq \mathbb{R}$ we define

$$m^*(S) = \inf \left\{ \sum_{I \in \mathcal{C}} \ell(I) \;\middle|\; \mathcal{C} = \text{collection of open intervals } I, \text{ so that } \bigcup_{I \in \mathcal{C}} I \supseteq S \right\}$$

The function $m^*: \mathcal{P}(\mathbb{R}) \to [0,\infty]$ is called the Lebesgue outer measure.

Proposition 32.3 The Lebesgue outer measure m^* has the following properties

1) $m^*(\emptyset) = 0$
2) For any $S, T \subseteq \mathbb{R}$ with $S \subseteq T$, $m^*(S) \le m^*(T)$
3) For any sequence $\{S_n\}_{n \in \mathbb{N}} \subseteq \mathcal{P}(\mathbb{R})$

$$(*) \quad m^*\left(\bigcup_{n \in \mathbb{N}} S_n\right) \le \sum_{n \in \mathbb{N}} m^*(S_n)$$

Proof (1) $\forall \varepsilon > 0$ $\mathcal{C} = \{(0,\varepsilon)\}$ is a collection of intervals (containing 1 interval). Since $\emptyset \subseteq (0,\varepsilon)$ $\forall \varepsilon$ and since $\ell((0,\varepsilon)) = \varepsilon$, $m^*(\emptyset) \le \varepsilon$ $\forall \varepsilon$.

(2) Suppose $\mathcal{C}$ is a collection of intervals with $T \subseteq \bigcup_{I \in \mathcal{C}} I$.
Then $S \subseteq T \subseteq \bigcup_{I \in \mathcal{C}} I$ $\Rightarrow$ (any cover of T by open intervals is a cover of S by open intervals) $\Rightarrow$

$$\Rightarrow \left\{ \sum_{I \in \mathcal{C}} \ell(I) \;\middle|\; \mathcal{C} \text{ a cover of } S \right\} \supseteq \left\{ \sum_{I \in \mathcal{C}} \ell(I) \;\middle|\; \mathcal{C} \text{ is a cover of } T \right\}$$

$$\Rightarrow m^*(S) = \inf(\text{LHS}) \le \inf(\text{RHS}) = m^*(T)$$

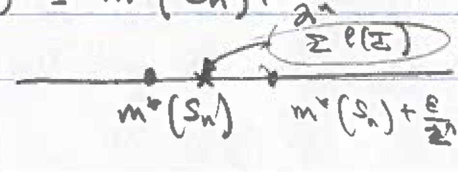
(3) $\forall k$, $S_k \subseteq \bigcup_{n \in \mathbb{N}} S_n$. Hence by (2)

$$m^*(S_k) \le m^*\left(\bigcup S_n\right)$$

Therefore if $m^*(S_k) = \infty$ for some k then $m^*(\bigcup S_n) = \infty$ as well and $(*)$ reduces to $\infty \le \infty$.

Next suppose $m^*(S_n) < \infty$ $\forall n$. Fix $\varepsilon > 0$.

$\forall n$ $\exists$ cover $\mathcal{C}_n$ of S_n s.t. $\displaystyle\sum_{I \in \mathcal{C}_n} \ell(I) \le m^*(S_n) + \frac{\varepsilon}{2^n}$



Let $\mathscr{C} = \bigcup_{n\in\mathbb{N}} \mathscr{C}_n$. Then $m^*(S) \leq \sum_{I\in\mathscr{C}} \ell(I) = \sum_{n\in\mathbb{N}} \sum_{I\in\mathscr{C}_n} \ell(I)$

$$\leq \sum_{n\in\mathbb{N}} \left(m^*(S_n) + \frac{\varepsilon}{2^n}\right) = \sum_{n\in\mathbb{N}} m^*(S_n) + \underbrace{\sum_{n=1}^{\infty} \frac{\varepsilon}{2^n}}_{=\varepsilon}$$

Since ε is arbitrary, we get.

$$m^*(S) \leq \sum_{n\in\mathbb{N}} m^*(S_n)$$

$\square$

<u>Definition</u> (Carathéodory's criterion) A set $E \subset \mathbb{R}$ is <u>Lebesgue measurable</u> if for any set $T \subset \mathbb{R}$

$$m^*(T\cap E) + m^*(T\cap E^c) = m^*(T)$$

If E is measurable we define the Lebesgue measure $m(E)$ of E to be the Lebesgue outer measure $m^*(E)$:

$$m(E) := m^*(E).$$

<u>Remark</u> 1) Since $\forall A, B \subset \mathbb{R}$ $m^*(A\cup B) \leq m^*(A) + m^*(B)$

and since $T = (T\cap E) \cup (T\cap E^c)$

$m^*(T) \leq m^*(T\cap E) + m^*(T\cap E^c)$

Thus E is Lebesgue measurable $\iff$ $\forall T \subseteq \mathbb{R}$,

$$m^*(T) \geq m^*(T\cap E) + m^*(T\cap E^c).$$

2) E is measurable $\iff$ E^c is measurable

3) $\emptyset$ is measurable: $\forall T$ $T\cap\emptyset = \emptyset$, $T\cap(\emptyset^c) = T\cap\mathbb{R} = T$

and $m^*(\emptyset) = 0$. Hence

$$m^*(T) = 0 + m^*(T) = m^*(T\cap\emptyset) + m^*(T\cap(\emptyset^c)).$$

<u>Proposition 32.4</u> If $E, F \subseteq \mathbb{R}$ are measurable then so is $E\cup F$ (hence, since $E\cap F = (E^c \cup F^c)^c$, $E\cap F$ is also measurable)