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<u>Recall</u> The <u>outer measure</u> $m^*(S)$ of $S \subseteq \mathbb{R}$ is

$$m^*(S) = \inf\left\{ \sum_{I \in \mathcal{C}} \ell(I) \;\middle|\; \mathcal{C} = \text{cover of } S \text{ by open intervals} \right\}$$

This gives us a function $m^*: \mathcal{P}(\mathbb{R}) \to [0, \infty]$ so that

<u>32.3</u> (1) $m^*(\emptyset) = 0$

(2) if $S \subseteq T$ then $m^*(S) \leq m^*(T)$

(3) $m^*\left(\bigcup_{n \in \mathbb{N}} S_n\right) \leq \sum_{n \in \mathbb{N}} m^*(S_n)$

<u>Def</u> A set $E \subseteq \mathbb{R}$ is <u>(Lebesgue) measurable</u> if $\forall T \subseteq \mathbb{R}$

$$m^*(E \cap T) + m^*(E^c \cap T) = m^*(T)$$

We saw:

- E is measurable $\iff$ $m^*(T) \geq m^*(E \cap T) + m^*(E^c \cap T)$
- $\emptyset$ is measurable
- E is measurable $\iff$ E^c is measurable

<u>Prop 32.4</u> If $E, F \subseteq \mathbb{R}$ are measurable then so is $E \cup F$.

<u>Proof</u> Let T be a set. Since E is measurable,

(1) $$m^*(T) = m^*(T \cap E) + m^*(T \cap E^c) \quad \text{and}$$

(2) $$m^*(\underbrace{T \cap (E \cup F)}_{T'}) = m^*((T \cap (E \cup F)) \cap E) + m^*((T \cap (E \cup F)) \cap E^c)$$
$$= m^*(T \cap E) + m^*(T \cap F \cap E^c)$$

Since F is measurable,

(3) $$m^*(T \cap E^c) = m^*((T \cap E^c) \cap F) + m^*((T \cap E^c) \cap F^c)$$

Therefore $$m^*(T) = \overbrace{\left(m^*(T \cap (E \cup F)) - \cancel{m^*(T \cap F \cap E^c)}\right)}^{m^*(T \cap E) \text{ from (2)}}$$
$$+ \underbrace{\cancel{m^*(T \cap E^c \cap F)} + m^*(T \cap E^c \cap F^c)}_{m^*(T \cap E^c) \text{ from (3)}}$$

$$= m^*(T \cap (E \cup F)) + m^*(T \cap (E \cup F)^c)$$

$\therefore$ $E \cup F$ is measurable

Cor 33.1. If E, F are measurable then so is $E \cap F$.

Proof $(E \cap F)^c = E^c \cup F^c$. E, F measurable $\Rightarrow E^c, F^c$ measurable

(32.4) $\Rightarrow$ $E^c \cup F^c$ is measurable. $\square$

Proposition 33.2 m is countably additive. That is if $\{E_k\}_{k \in \mathbb{N}}$ is a collection of measurable sets with $E_i \cap E_j = \emptyset$ for $i \neq j$ then

$\bigcup_k E_k = \coprod_k E_k$ is measurable and

$$m\left(\coprod_k E_k\right) = \sum_k m(E_k).$$

Proof We need to show: $\forall\, T \subseteq \mathbb{R}$

$$m^*(T) \geq m^*(T \cap U) + m^*(T \cap U^c) \quad \text{where } U := \bigcup_k E_k.$$

For each $n \in \mathbb{N}$ let

$$U_n := \bigcup_{k=1}^{n} E_k = \coprod_{k=1}^{n} E_k$$

By 32.4 (and induction) each U_n is measurable, so

$$m^*(T) = m^*(T \cap U_n) + m^*(T \cap (U_n)^c)$$

Since $U_n \subseteq U$, $(U_n)^c \supseteq U^c$. $\Rightarrow$ $T \cap (U_n)^c \supseteq T \cap U^c \Rightarrow m^*(T \cap (U_n)^c) \geq m^*(T \cap U^c)$

$\Rightarrow$ $m^*(T) \geq m^*(T \cap U_n) + m^*(T \cap U^c)$

Claim $m^*(T \cap U_n) \xrightarrow[n \to \infty]{} m^*(T \cap U)$.

Proof of claim:

$$m^*(T \cap U_k) = m^*((T \cap U_k) \cap E_k) + m^*(T \cap U_k \cap E_k^c) \quad (\text{since } E_k \text{ is measurable})$$
$$= m^*(T \cap E_k) + m^*(T \cap U_{k-1})$$

Induction on k $\Rightarrow$

$$m^*(T \cap U_n) = \sum_{k=1}^{n} m^*(T \cap E_k)$$

Next observe

$$\sum_{k=1}^{n} m^*(T \cap E_k) = m^*(T \cap U_n) \leq m^*(T \cap U) = m^*\left(\bigcup_{k=1}^{\infty} (T \cap E_k)\right)$$
$$\leq \sum_{k=1}^{\infty} m^*(T \cap E_k)$$

$$\Rightarrow \sum_{k=1}^{\infty} m^*(T \cap E_k) = \lim_{n \to \infty} \underbrace{\left(\sum_{k=1}^{n} m^*(T \cap E_k)\right)}_{m^*(T \cap U_n)} \leq m^*(T \cap U) \leq \sum_{k=1}^{\infty} m^*(T \cap E_k)$$

It follows $\lim_{n\to\infty} m^*(T\cap U_n) = m^*(T\cap U)$, which proves the claim.

Since $m^*(T) \geq m^*(T\cap U_n) + m^*(T\cap U^c)$

and since $\lim_{n\to\infty} m^*(T\cap U_n) = m^*(T\cap U)$

we get $m^*(T) \geq m^*(T\cap U) + m^*(T\cap U^c)$

$\Rightarrow$ U is measurable.

Finally let $T = \mathbb{R}$. Then

$$m(\underbrace{\bigcup E_k}_{U}) = m(U\cap\mathbb{R}) = \lim_{n\to\infty} m^*(U_n\cap\mathbb{R}) = \lim_{n\to\infty} \sum_{k=1}^{n} \underbrace{m^*(E_k\cap\mathbb{R})}_{=E_k}$$

$$= \sum_{k=1}^{\infty} m^*(E_k)$$

Corollary 33.3 For any sequence $\{E_k\}_{k=1}^{\infty}$ of measurable sets the union $\bigcup_{k\in\mathbb{N}} E_k$ is measurable.

Proof

Let $U_n = \bigcup_{k=1}^{n} E_k$, $F_1 = U_1$, $F_2 = U_2 \setminus U_1$, ... $F_n = U_n \setminus U_{n-1}$.

By 32.4 (and induction) U_n's are measurable.

Since $F_n = U_n \setminus U_{n-1} = U_n \cap (U_{n-1})^c$, F_n's are measurable

Also $F_i \cap F_j = \emptyset$ for $i \neq j$ and

$\bigcup_{i=1}^{\infty} F_i = \bigcup_{j=1}^{\infty} E_j$

By 33.2 $\bigcup F_i$ is measurable. $\Rightarrow \bigcup_{j=1}^{\infty} E_j$ is measurable.

Remains to prove: intervals are measurable and for an interval I $m(I) = \ell(I)$

Aside In the course we're following an interval is a connected bounded subset of $\mathbb{R}$, i.e. a set of the form $[a,b]$ or $(a,b]$ or $[a,b)$ or (a,b). Similarly a ray is a set of the form $[a,+\infty)$, $(a,+\infty)$, $(-\infty,b]$, $(-\infty,b)$.

<u>Note</u> Any interval is an intersection of two rays.

To prove that intervals are measurable it's enough to prove

<u>Proposition 34.4</u> A ray $R \subseteq \mathbb{R}$ is measurable.

<u>Proof</u> We need to show that $\forall T \subseteq \mathbb{R}$

$$(*) \quad m^*(T) \geq m^*(T \cap R) + m^*(T \cap R^c)$$

If $m^*(T) = \infty$, There is nothing to prove. So suppose $m^*(T) < \infty$.

Then $\forall \varepsilon > 0$ $\exists$ a cover $\mathcal{C}$ of T by open intervals so that

$$\sum_{I \in \mathcal{C}} \ell(I) \leq m^*(T) + \varepsilon/2.$$

Since $\sum_{I \in \mathcal{C}} \ell(I)$ is finite, $\ell(I) \neq 0$ only for countably many I.

If $I \in \mathcal{C}$ and $\ell(I) = \emptyset$, $I = \emptyset$. Discard empty elements of $\mathcal{C}$.

Therefore we may assume: $\mathcal{C}$ is countable. If $\mathcal{C}$ is finite $\mathcal{C} = \{I_1 \ldots I_N\}$ for some N. Otherwise $\mathcal{C} = \{I_n\}_{n=1}^{\infty}$.

$\forall n$, $I_n \cap R$, $I_n \cap R^c$ are intervals and

$$\ell(I_n \cap R) + \ell(I_n \cap R^c) = \ell(I_n).$$

For each n choose <u>open</u> intervals J_n, K_n so that

$I_n \cap R \subseteq J_n$, $I_n \cap R^c \subseteq K_n$ and $\ell(J_n) \leq \ell(I_n \cap R) + \varepsilon/2^{n+2}$

$\ell(K_n) \leq \ell(I_n \cap R^c) + \varepsilon/2^{n+2}$.

Then $\{J_n\}$ is an open cover of $T \cap R$, $\{K_n\}$ open cover of $T \cap R^c$.

$$\Rightarrow m^*(T \cap R) + m^*(T \cap R^c) \leq \sum_{n \in \mathbb{N}} \ell(J_n) + \sum_{n \in \mathbb{N}} \ell(K_n)$$

$$\leq \sum_{n=1}^{\infty} \left(\ell(I_n \cap R) + \frac{\varepsilon}{2^{n+2}} \right) + \sum_{n=1}^{\infty} \ell(I_n \cap R^c) + \frac{\varepsilon}{2^{n+2}}$$

$$= \sum_{n=1}^{\infty} \left(\ell(I_n \cap R) + \ell(I_n \cap R^c) \right) + \varepsilon \sum_{n=1}^{\infty} \frac{1}{2^{n+2}}$$

$$\leq \sum_{n=1}^{\infty} \ell(I_n) + \varepsilon/2 \leq m^*(T) + \varepsilon/2 + \varepsilon/2.$$

$\square$

$\therefore$ $(*)$ holds $\forall T$ and R is measurable.