Recall we have outer measure $m^*: P(\mathbb{R}) \to [0,\infty]$ with $S \subseteq T \Rightarrow m^*(S) \leq m^*(T)$

We defined a set $E \subseteq \mathbb{R}$ to be measurable if $\forall T \subseteq \mathbb{R}$

$$(*) \quad m^*(E \cap T) + m^*(E^c \cap T) = m^*(T)$$

(Note that $m^*(E \cap T) + m^*(E^c \cap T) \geq m^*((E\cap T) \cup (E^c \cap T)) = m^*(T)$ for any $E, T \subseteq \mathbb{R}$, so $(*)$ is equivalent to

$$(**) \quad m^*(T) \geq m^*(E \cap T) + m^*(E^c \cap T)$$

For a measurable set E we defined its measure $m(E)$ to be its outer measure: $m(E) := m^*(E)$

we proved:

Prop 33.2: countable unions of measurable sets are measurable. Moreover, if $\{E_n\}_{n=1}^{\infty}$ are measurable and pair-wise disjoint then

$$m(\coprod E_n) = \sum_{n=1}^{\infty} m(E_n) = \sum_{n \in \mathbb{N}} m(E_n)$$

Prop 34.0

Amusing fact: if $m^*(E) = 0$ then E is measurable (and $m(E) = 0$)

Proof $m^*(E \cap T) \leq m^*(E) = 0$. Hence $m^*(E \cap T) = 0$

Since $m^*(T) \geq m^*(T \cap E^c) = m^*(T \cap E^c) + 0 = m^*(T \cap E^c) + m^*(E \cap T)$

E is measurable.

Corollary Countable sets are measurable and have measure 0.

Proof: Sets of the form $\{x\}$, $x \in \mathbb{R}$ have outer measure 0.

If $E = \{x_i\}_{i \in \mathbb{N}}$ then $E = \coprod_{i \in \mathbb{N}} \{x_i\}$ and then, by 33.2

E is measurable and $m(E) = \sum_{i \in \mathbb{N}} m(\{x_i\}) = 0$.

Remark There are sets of measure 0 that are not countable.

See Cantor sets.

Recall I is an interval $\Rightarrow$ I is (a,b) or $[a,b)$ or $(a,b]$ or $[a,b]$

Similarly a ray is a connected set of the form (a,∞) or $[a,\infty)$ or $(-\infty,b)$ or $(-\infty,b]$.

Lemma 34.1 A ray $R \subseteq \mathbb{R}$ is measurable.

Proof We need to show that

$$m^*(T) \geq m^*(T\cap R) + m^*(T\cap R^c) \quad \forall\, T \subseteq \mathbb{R}.$$

If $m^*(T) = \infty$, we're done. So suppose $m^*(T) < \infty$.

Then $\forall \varepsilon > 0$, $\exists$ cover $\mathcal{C}$ of T by open intervals so that

$$\sum_{I\in\mathcal{C}} \ell(I) \leq m^*(T) + \varepsilon/2.$$

(since $m^*(T) = \inf_{\mathcal{C}'}\left(\sum_{I\in\mathcal{C}'} \ell(I)\right)$).

Since $\sum_{I\in\mathcal{C}} \ell(I)$ is finite, $\ell(I)\neq 0$ for only countably many I, see lecture on sums over sets.

$\ell(I)=0 \Rightarrow I=\emptyset$ (since I is open). Therefore it's no loss of generality to assume that $\mathcal{C}$ is countable (discard empty sets)

If $\mathcal{C}$ is finite, $\mathcal{C} = \{I_1, \dots, I_N\}$ for some N.

Otherwise $\mathcal{C} = \{I_n\}_{n=1}^{\infty}$. We consider this case.

$\forall n$, $I_n\cap R$, $I_n \cap R^c$ are intervals (not necessarily open) and

$$\ell(I_n\cap R) + \ell(I_n\cap R^c) = \ell(I_n).$$

(Note that I am using ℓ = length, not m^* = outer measure)

For each n choose open intervals J_n, K_n so that

$$I_n\cap R \subseteq J_n, \quad I_n \cap R^c \subseteq K_n \text{ and}$$

$$\ell(J_n) \leq \ell(I_n\cap R) + \varepsilon/2^{n+2} \qquad \ell(K_n) \leq \ell(I_n\cap R^c) + \varepsilon/2^{n+2}$$

Then $\{J_n\}_{n\in\mathbb{N}}$ is an open cover of $T\cap R$, $\{J_n\}$ an open cover of $T\cap R^c$.

$\Rightarrow$

$$\begin{aligned}
m^*(T\cap R) + m^*(T\cap R^c) &\leq \sum_{n\in\mathbb{N}} \ell(J_n) + \sum_{n\in\mathbb{N}} \ell(K_n) \\
&\leq \sum_{n=1}^{\infty}\left(\ell(I_n\cap R) + \frac{\varepsilon}{2^{n+2}}\right) + \sum_{n=1}^{\infty}\left(\ell(I_n\cap R^c) + \frac{\varepsilon}{2^{n+2}}\right) \\
&= \sum_{n=1}^{\infty}\left(\ell(I_n\cap R) + \ell(I_n\cap R^c)\right) + \varepsilon\sum_{n=1}^{\infty}\frac{1}{2^{n+2}} \\
&\leq \sum_{n=1}^{\infty} \ell(I_n) + \varepsilon\cdot\frac{1}{2} \leq (m^*(T) + \varepsilon/2) + \varepsilon/2
\end{aligned}$$

Since ε is arbitrary, $m^*(T \cap R) + m^*(T \cap R^c) \leq m^*(T)$.

$\Rightarrow$ R is measurable.

<u>Corollary</u> 34.2 Intervals are measurable.

<u>Proof</u> $(a,b) = (-\infty, b) \cap (a, +\infty)$ etc.

Remains to prove:

<u>Theorem 34.3</u> $\ell(J) = m(J)$ $(= m^*(J))$

for any interval $J \subseteq \mathbb{R}$.

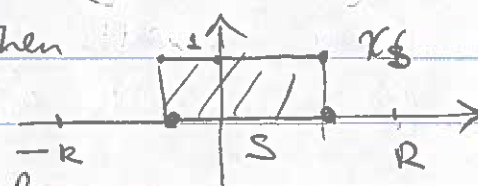
Belk's proof is a bit of a cheat: it uses Darboux integrals

<u>Notation</u> For a subset $S \subseteq \mathbb{R}$, $\chi_S : \mathbb{R} \to [0,1]$ is defined by

$$\chi_S(x) = \begin{cases} 1 & x \in S \\ 0 & x \notin S. \end{cases}$$

<u>Note</u> If $S \subset \mathbb{R}$ is an interval χ_S is integrable on $[a,b]$ for any a, b. And if $S \subseteq [-R, R]$ then

$$\int_{-R}^{R} \chi_S = \ell(S)$$



In particular $\int_{-\infty}^{\infty} \chi_S$ exists and equals $\int_{-R}^{R} \chi_S = \ell(S)$ for large enough R.

<u>Proof of 34.3</u> : We first argue that $m(J) \leq \ell(J)$:

$\forall \varepsilon > 0$ $\exists$ open interval J' so that $J \subseteq J'$ and $\ell(J') = \ell(J) + \varepsilon$.

$\mathcal{C}_0 = \{J'\}$ is an open cover of J.

$$\Rightarrow \quad m(J) = \inf_{\mathcal{C}} \{ \sum_{I \in \mathcal{C}} \ell(I) \} \leq \ell(J') = \ell(J) + \varepsilon.$$

Since ε is arbitrary,

$$m(J) \leq \ell(J).$$

Now let $\mathcal{C}$ be a collection of open intervals with $J \subseteq \bigcup_{I \in \mathcal{C}} I$

For any $\varepsilon > 0$ $\exists$ closed interval K s.t. $K \subseteq J$ and $\ell(K) \geq \ell(J) - \varepsilon$

Since K is compact, and since $\mathcal{C}$ covers K, $\exists$ finite subcover $\{I_1, \dots I_n\} \subseteq \mathcal{C}$ of K. Then $K \subseteq I_1 \cup \dots \cup I_n$

$$\Rightarrow \chi_K \leq \chi_{I_1} + \dots + \chi_{I_n}$$

$$\Rightarrow \ell(K) = \int_{-\infty}^{\infty} \chi_K \leq \int_{-\infty}^{\infty} (\chi_{I_1} + \dots + I_n) = \int_{-\infty}^{\infty} \chi_{I_1} + \dots + \int_{-\infty}^{\infty} I_n = \ell(I_1) + \dots + \ell(I_n) \leq \sum_{I \in \mathcal{C}} \ell(I)$$

$$\Rightarrow \ell(J) - \varepsilon \leq \sum_{I \in \mathcal{C}} \ell(I) \quad \forall \mathcal{C}$$

$$\Rightarrow \ell(J) - \varepsilon \leq m^*(J) = \inf_{\mathcal{C}} \sum_{I \in \mathcal{C}} \ell(I) = m(J)$$

Since ε is arbitrary $\ell(J) \leq m(J)$. ∎

We are now almost ready to define Lebesgue integrals.
But we won't be able to integrate all functions.

<u>Definition</u> A function $f : \mathbb{R} \to [-\infty, \infty]$ is <u>measurable</u> if the sets $f^{-1}((a, \infty]) = \{x \in \mathbb{R} \mid f(x) > a\}$ are measurable for all $a \in \mathbb{R}$.

If $f : \mathbb{R} \to [-\infty, \infty]$ is measurable we'll define $\int_{\mathbb{R}} f$ in two steps.
First we define $\int_{\mathbb{R}} f$ if $f(x) \geq 0$ $\forall x$
Then given an arbitrary $f : \mathbb{R} \to [-\infty, \infty]$

we set $f^+(x) = \begin{cases} f(x) & f(x) \geq 0 \\ 0 & f(x) \leq 0 \end{cases}$, $f^-(x) = \begin{cases} -f(x) & f(x) \leq 0 \\ 0 & f(x) \geq 0 \end{cases}$

So that $f(x) = f^+(x) - f^-(x)$

And then

$$\int_{\mathbb{R}} f = \int_{\mathbb{R}} f^+$$