4/22/2024

MATH424 Eugene Lerman

35:1

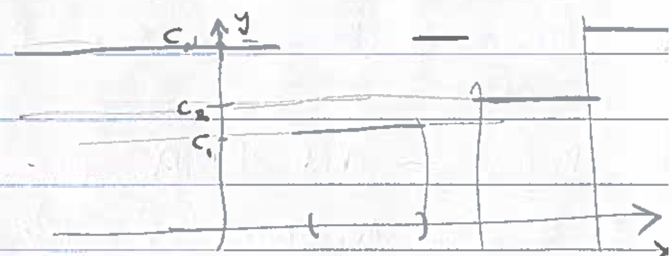
Last time • (Outer) measure 0 sets are measurable
• rays and intervals are measurable and $m((a,b)) = \underbrace{b-a}_{\ell((a,b))}$ $\forall a \le b$.

We thus have a set $\mathbb{R}$, a set $\mathcal{M} \subseteq \mathcal{P}(\mathbb{R})$ of Lebesgue measurable sets and a function $m : \mathcal{M} \to [0,\infty]$, Lebesgue measure with
$$m((a,b)) = b-a$$
The triple $(\mathbb{R}, \mathcal{M}, m: \mathcal{M} \to [0,\infty])$ is an example of a measure space
(And I haven't defined genral measure spaces nor general measures)

Recall: A function $f : \mathbb{R} \to [-\infty, \infty]$ is measurable if the sets $\{x \in \mathbb{R} \mid f(x) > a\}$ are measurable for all $a \in \mathbb{R}$.

Def A function $s : \mathbb{R} \to \mathbb{R}$ is simple if it's measurable and takes only finitely many values.

This amounts to $\exists k$, $c_1 \ldots c_N \in \mathbb{R}$ so that
$$s = \sum_{n=1}^{N} c_n \chi_{E_n} \qquad \text{where } E_n := s^{-1}(c_n) \ \forall n$$
are measurable sets
with $\mathbb{R} = \coprod_{n=1}^{N} E_n$



Definition Let $s : \mathbb{R} \to [0,\infty)$ be a non-negative simple function, $E \subseteq \mathbb{R}$ measurable set. We define the integral $I_E(s)$ of s over E by
$$I_E(s) = \sum_{i=1}^{N} c_i \, m(E \cap E_i)$$
(where $c_1 \ldots c_N$ are the values of s and $E_i = s^{-1}(c_i)$ $\forall i$).

Remark

$I_E(s)$ can be $+\infty$ since $m(E \cap E_i)$ can be $+\infty$.

<u>Proposition 35.1</u> The integral $I_E : \{\text{non-neg.-simple functions}\} \to [0,\infty]$ is linear and monotone:

a. $I_E(cs) = c\, I_E(s) \quad \forall c \geq 0,\ \forall s$

b. $I_E(s_1+s_2) = I_E(s_1) + I_E(s_2) \quad \forall s_1, s_2$

c. if $s_1 \leq s_2$ then $I_E(s_1) \leq I_E(s_2)$.

<u>Proof</u> a. exercise.

b. Let $c_1 .. c_m$ be the distinct values of s_1, $d_1 .. d_n$ the distinct values of s_2

$E_i = s_1^{-1}(c_i)$, $F_j = s_2^{-1}(d_j)$.

Then $\mathbb{R} = \coprod_i E_i = \coprod_j F_j$ and $(s_1+s_2)(x) = c_i + d_j \quad \forall x \in E_i \cap F_j$ $\forall i,j$. $\Rightarrow$

$$I_E(s_1+s_2) = \sum_{i,j} (c_i+d_j)\, m(E_i \cap F_j \cap E)$$

$$= \sum_i c_i \cdot \sum_j m((E_i \cap F_j) \cap E) + \sum_j d_j \cdot \sum_i m(E_i \cap F_j \cap E)$$

$$= \sum_i c_i\, m(E_i \cap E) + \sum d_j\, m(F_j \cap E)$$

$$= I_E(s_1) + I_E(s_2).$$

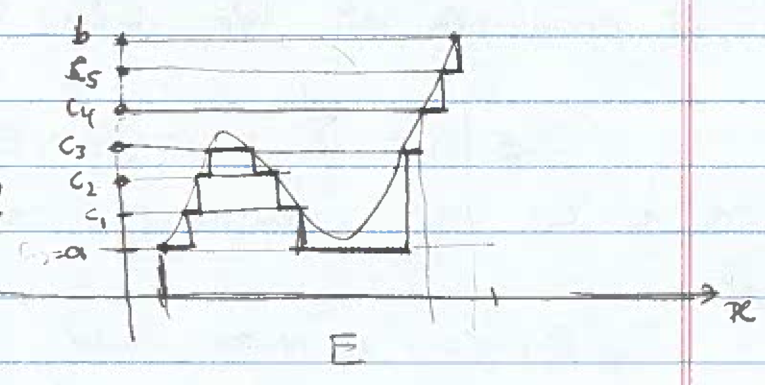
c. $s_2 - s_1$ is a nonnegative simple function and $s_2 = (s_2 - s_1) + s_1$

Hence by (b) $I_E(s_2) = I_E(s_2 - s_1) + I_E(s_1) \geq I_E(s_1)$. $\square$

<u>Definition</u> Let $f : \mathbb{R} \to [0,\infty]$ be a nonnegative measurable function, $E \subseteq \mathbb{R}$ measurable. We define

$$\int_E f\, dm := \sup\{ I_E(s) \mid 0 \leq s \leq f,\ s \text{ simple} \}.$$

<u>Picture</u>: say $f(E) = [a,b]$



We are partitioning the <u>range</u> of f so that the preimages of these pieces are measurable.

<u>Proposition 35.2</u> Let $s: \mathbb{R} \to [0,\infty]$ be a nonnegative simple function. Then

$$I_E(s) = \int_E s\, dm.$$

<u>Proof</u> Since $s \le s$, $I_E(s) \le \sup\{ I_E(s') \mid s' \le s\} = \int_E s\, dm$.
On the other hand $\forall$ simple function s' with $s' \le s$, $I_E(s') \le I_E(s)$ by 35.1(c). Hence $\int_E s\, dm = \sup\{ I_E(s') \mid s' \le s\} \le I_E(s)$.

$\square$

<u>Theorem 35.3</u> Let $f: \mathbb{R} \to [0,\infty]$ be measurable. There is a sequence of non-negative simple functions

$$0 \le s_1 \le s_2 \cdots \le f$$

so that $s_n \to f$ pointwise. If f is bounded then $s_i \to f$ uniformly.

<u>Proof</u> We first define s_n $\forall n$. Consider $[0,n) \subseteq \mathbb{R}$.

Let $I_i = \{ t \in \mathbb{R} \mid \frac{i-1}{2^n} \le t < \frac{i}{2^n} \}$, $1 \le i \le n2^n$.

Let $E_i = f^{-1}(I_i)$, $F_n = f^{-1}([n, +\infty])$.

Then $\mathbb{R} = F_n \sqcup \left(\bigsqcup_{1 \le i \le n2^n} f^{-1}(I_i) \right)$

Define $s_n(x) = \sum_{i=1}^{n2^n} \left(\frac{i-1}{2^n}\right) \chi_{E_i}(x) + n\chi_{F_n}$

Then $\forall x \in E_i$

$$\frac{i-1}{2^n} \le f(x) < \frac{i}{2^n}, \quad s_n(x) = \frac{i-1}{2^n}.$$

$$\Rightarrow \quad s_n(x) \le f(x) \quad \forall x \in E_i, \ i = 1, \ldots n2^n$$

Similarly $\forall x \in F_n$, $s_n(x) = n$ and $n \le f(x)$

$$\Rightarrow \quad s_n(x) \le f(x) \quad \forall x \in F_n.$$

Hence $s_n(x) \le f(x)$ $\forall x \in \mathbb{R}$.

<u>Claim</u> $s_n \le s_{n+1}$.

<u>Proof of claim</u>: $\underset{(i \le n2^n)}{\forall i}$, $\underbrace{\left[\frac{i-1}{2^n}, \frac{i}{2^n}\right)}_{I} = \underbrace{\left[\frac{2i-2}{2^{n+1}}, \frac{2i-1}{2^{n+1}}\right)}_{I'} \sqcup \underbrace{\left[\frac{2i-1}{2^{n+1}}, \frac{2i}{2^{n+1}}\right)}_{I''}$.

Let $E = f^{-1}(I)$, $E' = f^{-1}(I')$, $E'' = f^{-1}(I'')$. Then

$$S_n(x) = \frac{i-1}{2^n} \quad \text{for } x \in E$$

$$S_{n+1}(x) = \frac{i-1}{2^n} \text{ for } x \in E', \qquad S_{n+1}(x) = \frac{2i-1}{2^{n+1}} \quad \text{for } x \in E''.$$

Since $E = E' \amalg E''$ and $\frac{i-1}{2^n} = \frac{2i-2}{2^{n+1}} < \frac{2i-1}{2^{n+1}}$

$$S_n(x) \le S_{n+1}(x) \quad \forall x \in E$$

Similar argument shows:

$$S_n(x) \le S_{n+1}(x) \quad \forall x \in f^{-1}([n,\infty)) \equiv F_n.$$

$$\therefore \quad S_n(x) \le S_{n+1}(x) \quad \forall x.$$

Next we argue $S_n(x) \xrightarrow[n\to\infty]{} f(x) \quad \forall x \in \mathbb{R}$

Case 1 $f(x) = +\infty$. Then $x \in F_n$, $\forall n \Rightarrow S_n(x) = n \ \forall n$ and $n \to +\infty$.

So $S_n(x) \xrightarrow[n\to\infty]{} f(x)$ if $f(x) = +\infty$.

Case 2 $f(x)$ is finite.

Say $f(x) < n_0$ for some $n_0 \in \mathbb{N}$. Then for $n > n_0$ $f(x) \notin [n, \infty)$

$$\Rightarrow \frac{i-1}{2^n} \le f(x) < \frac{i}{2^n} \quad \text{for some } i$$

$$\Rightarrow S_n(x) = \frac{i-1}{2^n}. \quad \text{So} \quad |f(x) - S_n(x)| < \frac{1}{2^n}. \quad \forall x$$

$$\Rightarrow S_n(x) \to f(x).$$

Moreover, since for $n > n_0$ $|f(x) - S_n(x)| < \frac{1}{2^n}$ for all x

$S_n \to f$ uniformly on $\mathbb{R}$.

$\square$