MATH 424 Eugene Lerman 4/24/2024

First: measurable functions with more care.

<u>Definition</u> $f: \mathbb{R} \to [-\infty, \infty]$ is measurable if $\forall a \in \mathbb{R}$ the set $\{x \in \mathbb{R} \mid f(x) > a\} = f^{-1}((a, \infty])$ is measurable.

<u>Proposition 36.1</u> Let $f: \mathbb{R} \to [-\infty, \infty]$ be a function. The following are equivalent:

(1) $f^{-1}((a, \infty])$ is measurable $\forall a \in \mathbb{R}$

(2) $f^{-1}([a, \infty])$ — // —

(3) $f^{-1}([-\infty, a))$ — // —

(4) $f^{-1}([-\infty, a])$ — // —

(5) The sets $f^{-1}(\{-\infty\})$, $f^{-1}(\{\infty\})$ and $f^{-1}((a,b))$ are measurable for all $a < b$.

<u>Corollary 36.2</u> If $f: \mathbb{R} \to [-\infty, \infty]$ is measurable Then $f^{-1}(a)$ is measurable $\forall a$. In particular simple functions are measurable.

<u>Proof of 36.2</u> $f^{-1}(a) = f^{-1}([-\infty, a]) \cap f^{-1}([a, \infty])$

<u>Proof of 36.1</u> (1) $\Rightarrow$ (2) $f^{-1}([a, \infty]) = \bigcap_{n=1}^{\infty} (a - \frac{1}{n}, \infty]$

(2) $\Rightarrow$ (3) $f^{-1}([-\infty, a)) = \mathbb{R} \setminus f^{-1}([a, \infty])$

(3) $\Rightarrow$ (4) $f^{-1}([-\infty, a]) = \bigcap_{n=1}^{\infty} f^{-1}([-\infty, a + 1/n))$

(4) $\Rightarrow$ (5)

We know that $\forall c$ $f^{-1}([-\infty, c])$ is measurable

Then $f^{-1}([-\infty, b)) = \bigcup_{n=1}^{\infty} f^{-1}([-\infty, b - 1/n])$, hence is measurable

$\Rightarrow f^{-1}((a,b)) = f^{-1}([-\infty, b)) \setminus f^{-1}([-\infty, a])$ is measurable, $\forall a, b$.

Moreover $f^{-1}(\{-\infty\}) = \bigcap_{n=1}^{\infty} f^{-1}([-\infty, -n])$

$f^{-1}(\{\infty\}) = \bigcup_{n=1}^{\infty} f^{-1}([-\infty, n])$, hence are measurable

$(5) \Rightarrow (1)$ $\quad f^{-1}((a,\infty]) = f^{-1}(\{\infty\}) \cup \bigcup_{n=1}^{\infty} f^{-1}(a, a+n)$

Recall if $s = \sum_{i=1}^{N} c_i \chi_{E_i}$ for some $E_1 \ldots E_N \subseteq \mathbb{R}$, measurable, $c_1 \ldots c_N \in \mathbb{R}$

Then for any measurable set E,

$$I_E(s) = \sum_{i=1}^{N} c_i\, m(E \cap E_i)$$

If $f: \mathbb{R} \to [0, \infty]$ is measurable, $E \subseteq \mathbb{R}$ measurable

$$\int_E f\, dm := \sup\{ I_E(s) \mid 0 \le s \le f, \ s \text{ simple}\}$$

We showed:

for a simple function s with $s \ge 0$

$$\int_E s\, dm = I_E(s).$$

We didn't finish proving

Thm 35.3 Suppose $f: [0,\infty]$ is measurable. Then there is a sequence of non-negative simple functions $0 \le s_1 \le s_2 \le \ldots \le f$ So that $s_n \to f$ pointwise. If f is bounded, then $s_n \to f$ uniformly.

Proof Recall that we defined $s_n = \sum_{i=1}^{n2^n - 1} \frac{i-1}{2^n} \chi_{E_i}(x) + \chi_{F_n}$

$$F_n = f^{-1}([n, \infty])$$

$$E_i = f^{-1}\left(\left[\frac{i-1}{2^n}, \frac{i}{2^n}\right)\right), \quad 1 \le i < n2^n$$

We checked:

$$s_n(x) \le f(x) \quad \forall x.$$

Claim 1 $\quad s_n(x) \le s_{n+1}(x) \quad \forall x$

Proof of claim:

For $1 \le i < n2^n$ $\quad \underbrace{\left[\frac{i-1}{2^n}, \frac{i}{2^n}\right)}_{I} = \underbrace{\left[\frac{2i-2}{2^{n+1}}, \frac{2i-1}{2^{n+1}}\right)}_{I'} \amalg \underbrace{\left[\frac{2i-1}{2^{n+1}}, \frac{2i}{2^{n+1}}\right)}_{I''}$

Let $E = f^{-1}(I)$, $\quad E' = f^{-1}(I')$, $\quad E'' = f^{-1}(I'')$

Then for $x \in E$, $\quad s_n(x) = \frac{i-1}{2^n}$

$$s_{n+1}(x) = \frac{i-1}{2^n} \text{ for } x \in I' \quad \text{and} \quad s_{n+1}(x) = \frac{2i-1}{2^{n+1}} \text{ for } x \in E''$$

Since $E = E' \amalg E''$ and since $\frac{i-1}{2^n} = \frac{2i-2}{2^{n+1}} < \frac{2i-1}{2^{n+1}}$

$$s_n(x) \le s_{n+1}(x) \quad \forall x \in E.$$

Similar argument shows

$$s_n(x) \le s_{n+1}(x) \quad \text{for } x \in f^{-1}([n,\infty)) = F_n$$

and claim 1 follows.

<u>Claim 2</u> $s_n(x) \xrightarrow[n\to\infty]{} f(x) \quad \forall x \in \mathbb{R}$

<u>Case 1</u> $f(x) = +\infty$. Then $x \in F_n \; \forall n. \Rightarrow s_n(x) = n \quad \forall n$

$$\Rightarrow n = s_n(x) \xrightarrow[n\to\infty]{} f(x) = \infty.$$

<u>Case 2</u> $f(x)$ is finite.

Then $f(x) < n_0$ for some $n_0 \in \mathbb{N}$. Then for $n > n_0$

$f(x) \notin [n,\infty)$, $\Rightarrow \frac{i-1}{2^n} \le f(x) < \frac{i}{2^n}$ for some i

Since $s_n(x) = \frac{i-1}{2^n}$,

$$|f(x) - s_n(x)| < \frac{1}{2^n} \quad \text{for all } n \ge n_0.$$

$$\Rightarrow s_n(x) \xrightarrow[n\to\infty]{} f(x).$$

Moreover since for $n > n_0$ $|f(x) - s_n(x)| < \frac{1}{2^n}$ for <u>all</u> x

$s_n \to f$ uniformly.

Adams - Guillemin p 64–65

<u>Proposition 36.2</u> (properties of the Lebesgue integral)

E, F measurable sets, f, g nonnegative measurable functions on $\mathbb{R}$

Then

1) If $f \le g$ then $\int_E f\,dm \le \int_E g\,dm$

2) If $E \subseteq F$ then $\int_E f\,dm \le \int_F f\,dm$

3) If $m(E) = 0$ Then

$$\int_E f\,dm = 0$$

<u>Proof</u> 1) Recall $\int_E f\,dm = \sup\{ I_E(s) \mid s \le f, \ s \text{ simple} \}$

Since $f \le g$ $\{s \text{ simple} \mid s \le f\} \subseteq \{s' \text{ simple} \mid s' \le g\}$

$$\Rightarrow \underbrace{\sup\{I_E(s) \mid s \le f\}}_{\int_E f} \le \underbrace{\sup\{I_E(s') \mid s' \le g\}}_{\int_E g}$$

2. If $f = \chi_G$, $G \subseteq \mathbb{R}$ measurable, then

$$(*) \quad \int_E \chi_G = m(G \cap E) \leq m(G \cap F) = \int_F \chi_G$$

$\Rightarrow$ If $f = \sum_{i=1}^{n} c_i \chi_{G_i}$, then $(**)$ $\int_E f = \sum c_i \int_E \chi_{G_i} \overset{(*)}{\leq} \sum c_i \int_F \chi_{G_i}$

$= \int_F f$.

Hence 2. is true for simple functions.

Finally for an arbitrary f

$$\int_E f = \sup \{ \int_E s \mid s \text{ simple}, s \leq f \}$$

$$\overset{(**)}{\leq} \sup \{ \int_F s \mid s \text{ simple}, s \leq f \} = \int_F f.$$

3. Same strategy of proof as above:

if $f = \chi_G$, $m(E) = 0$, then $\int_E \chi_G = m(G \cap E) \leq m(E) = 0$.

$\Rightarrow \int_E f = 0$ in this case

if $f = \sum c_i \chi_{G_i}$,

$$\int_E f = \sum c_i \int_E \chi_{G_i} = \sum c_i \cdot 0 = 0$$

Finally for a general function f

$$\int_E f = \sup \{ \int_E s \mid 0 \leq s \leq f, \ s \text{ simple} \}$$

$$= \sup \{ 0 \mid 0 \leq s \leq f \} = 0.$$

$\square$