MATH 424 Eugene Lerman

Last time: Finished proving: $f: \mathbb{R} \to [0,\infty]$ measurable function
Then $\exists$ sequence $s_1 \le s_2 \le \dots \le s_n \le \dots \le f$
of non-negative simple functions so that $s_n \to f$ pointwise
If f is bounded $s_n \to f$ uniformly.

We also proved: if $0 \le f \le g$ and $E \subseteq F$ measurable, then

(i) $\int_E f\, dm \le \int_E g$

(ii) $\int_E f\, dm \le \int_F f\, dm$

(iii) If $m(E) = 0$ then $\int_E f\, dm = 0$.

Aside: Arithmetic on $[-\infty, +\infty]$:

for $x \in \mathbb{R}$ $\quad x + (\pm\infty) = \pm\infty, \quad x - (\pm\infty) = \mp\infty$

$(+\infty) + (+\infty) = +\infty = (+\infty) - (-\infty)$

But $(+\infty) + (-\infty)$, $(+\infty) - (+\infty)$ and $(-\infty) - (-\infty)$ are not defined.

Thus for $x, y \in [-\infty, +\infty]$, $x+y$ is defined if $\{x,y\} \ne \{+\infty, -\infty\}$

Multiplication:

$$(\pm\infty)\cdot(\pm\infty) = +\infty, \quad (\pm\infty)\cdot(\mp\infty) = -\infty$$

For $x \in \mathbb{R}$ $\quad x\cdot(\pm\infty) = (\pm\infty)\cdot x = \begin{cases} \pm\infty & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ \mp\infty & \text{for } x < 0. \end{cases}$

Definition: Given $f: \mathbb{R} \to [-\infty, \infty]$ we define

$$f_+(x) = \begin{cases} f(x) & f(x) \ge 0 \\ 0 & f(x) \le 0 \end{cases} \qquad f_-(x) = \begin{cases} -f(x) & f(x) \le 0 \\ 0 & f(x) \ge 0 \end{cases}$$

Then $f = f_+ - f_-$

Lemma 37.1 If $f: \mathbb{R} \to [-\infty, \infty]$ is measurable then so are f_+ and f_-

Proof For $a > 0$, $(f_+)^{-1}((a,\infty]) = f^{-1}((a,\infty])$

for $a<0$, $(f_+)^{-1}((a,\infty]) = \mathbb{R}$.

Similarly for $a<0$ $(f_-)^{-1}((a,\infty]) = \mathbb{R}$

For $a>0$, $x \in (f_-)^{-1}((a,\infty]) \Leftrightarrow -f(x) \in (a,\infty]$

$\Leftrightarrow f(x) \in [-\infty, -a) \Leftrightarrow x \in f^{-1}([-\infty,-a))$

Thus $(f_-)^{-1}((a,\infty]) = [-\infty, -a)$. $\square$

__Definition__ Let $f : \mathbb{R} \to [-\infty,\infty]$ be a measurable function, $E \subseteq \mathbb{R}$ measurable set. Suppose $\int_E f_+ \, dm$, $\int_E f_- \, dm$ are finite. Then we define

$$\int_E f \, dm := \int_E f_+ \, dm - \int_E f_- \, dm$$

__Aside__ It may happen that $\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R_1,R_2\to\infty} \int_{-R_1}^{R_2} f(x)\,dx$ exists, but $\int_{\mathbb{R}} f_+ \, dm$, $\int_{\mathbb{R}} f_- \, dm$ are $+\infty$ hence $\int_{\mathbb{R}} f \, dm$ does not exist.

__Ex__ $f(x) = \begin{cases} \frac{(-1)^n}{n} & n-1 \le x < n \quad n \in \mathbb{N}. \\ 0 & x < 0 \end{cases}$

Then $\int_{-\infty}^{\infty} f(x)\,dx = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, which exists.

But $\int_{\mathbb{R}} f_+ \, dm = \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{2n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} = +\infty$

and similarly

$$\int_{\mathbb{R}} f_- \, dm = +\sum_{n=1}^{\infty} \frac{(-1)^{2n}}{2n+1} = +\infty.$$

__Note__ $|f| = f_+ + f_-$. If f measurable, is $|f|$ measurable

Proposition 37.2 Suppose $f, g : \mathbb{R} \to [-\infty, \infty]$ are measurable and $f+g$ is defined (ie $\nexists x$ s.t. $f(x) = \pm\infty$ $g(x) = -f(x)$)
Then $f+g$ is measurable.

Proof $\forall\, a \in \mathbb{R}$

$$\{x \mid f(x) + g(x) > a\} = \{x \mid f(x) > a - g(x)\}$$
$$= \{x \mid f(x) > r > a - g(x) \text{ for some } r \in \mathbb{Q}\}$$
$$= \bigcup_{r \in \mathbb{Q}} \left(\{x \mid f(x) > r\} \cap \{x \mid a - g(x) < r\} \right),$$

which is a countable union of measurable sets.

Cor if f is measurable then so is $|f|$.

Exercise Let $0 \le f_1 \le f_2 \le \dots \le f_n$ be a sequence of measurable functions, $f(x) = \lim_{n \to \infty} f_n(x)$. Then f is measurable.

Lemma 37.3 Let $s : \mathbb{R} \to [0, \infty)$ be a simple function, $E_1 \subseteq E_2 \subseteq \dots \subseteq E_n \subseteq \dots$ a sequence of measurable sets, $E = \bigcup_{n=1}^{\infty} E_n$. Then

$$\int_E s\, dm = \lim_{n \to \infty} \int_{E_n} s\, dm.$$

Proof

As we have seen in lecture 36, it's no loss of generality to assume that $s = \chi_G$ for some measurable G. Then

$$\int_{E_n} s\, dm = m(G \cap E_n) \ \forall n; \quad \int_E s\, dm = m(G \cap E)$$

Since

$$G \cap E_1 \subseteq G \cap E_2 \subseteq \dots \subseteq G \cap E_n \subseteq \dots \quad \text{and} \quad \bigcup_{n=1}^{\infty} (G \cap E_n) = G \cap E,$$
$$m(G \cap E) = \lim_{n \to \infty} m(G \cap E_n).$$

We're done.

Remark Result holds for arbitrary functions, See Adams-Guillemin, p. 67.

Lebesgue's monotone convergence Theorem

Let $0 \le f_1 \le f_2 \le \cdots \le f_n \le \cdots$ be a sequence of measurable functions, $f = \lim_{n\to\infty} f_n$. Then

$$\lim_{n\to\infty}\left(\int_{\mathbb{R}} f_n \, dm\right) = \int_{\mathbb{R}} f \, dm \left(= \int_{\mathbb{R}} (\lim f_n) \, dm\right)$$

Proof Since $f_n \le f \ \forall n$, $\int_{\mathbb{R}} f_n \, dm \le \int_{\mathbb{R}} f \, dm \ \forall n$.

$\Rightarrow \lim_{n\to\infty} \int_{\mathbb{R}} f_n \, dm \le \int_{\mathbb{R}} f \, dm$.

To prove that

$$\int_{\mathbb{R}} f \, dm \ge \lim_{n\to\infty} \int_{\mathbb{R}} f_n \, dm,$$

Consider a simple function s with $0 \le s \le f$.

We argue: (*) $\lim_{n\to\infty} \int_{\mathbb{R}} f_n \, dm \ge \int_{\mathbb{R}} s \, dm$.

Choose $\varepsilon > 0$ and let

$$E_n = \{x \in \mathbb{R} \mid f_n(x) > (1-\varepsilon) s\} \quad \forall n.$$

Note that $E_n = \{x \in \mathbb{R} \mid f_n(x) + (\varepsilon - 1) \cdot s > 0\}$ which is measurable since f_n and $(\varepsilon - 1) \cdot s$ are measurable functions. Also $\bigcup_{n=1}^{\infty} E_n = \mathbb{R}$ since $f_n \nearrow f$, and $f \ge s$.

$$\int_{\mathbb{R}} f_n \, dm \ge \int_{E_n} f_n \, dm \ge (1-\varepsilon) \int_{E_n} s \, dm.$$

By 37.3, $\lim_{n\to\infty} (1-\varepsilon) \int_{E_n} s \, dm = (1-\varepsilon) \int_{\mathbb{R}} s \, dm$.

$\Rightarrow \lim_{n\to\infty} \int_{\mathbb{R}} f_n \, dm \ge \lim_{n\to\infty} (1-\varepsilon) \int_{E_n} s \, dm = (1-\varepsilon) \int_{\mathbb{R}} s \, dm$.

Since ε is arbitrary $\lim_{n\to\infty} \int_{\mathbb{R}} f_n \, dm \ge \int_{\mathbb{R}} s \, dm$ and we're done.