MATH 424 Eugene Lerman 4/29/2024

Last time: Monotone convergence Theorem:

$0 \le f_1 \le f_2 \le \ldots \le f_n \le \ldots$ sequence of measurable functions

$f = \lim_{n\to\infty} f_n$, $E \subseteq \mathbb{R}$ measurable. Then

$$\lim_{n\to\infty} \int_E f_n \, dm = \int_E f \, dm.$$

Thm 38.1 $f, g : \mathbb{R} \to [0,\infty]$ two measurable functions, $c > 0$, E measurable set. Then

(1) $\int_E (c f) \, dm = c \int_E f \, dm$

(2) $\int_E (f+g) \, dm = \int_E f \, dm + \int_E g \, dm$

Proof (1). Suppose $s : \mathbb{R} \to [0,\infty]$ is simple. Then $I_E(cs) = c\, I_E(s)$ and $s \le f \iff cs \le cf$.

Therefore

$$\begin{aligned} c \int_E f \, dm &= c \sup\{ I_E(s) \mid 0 \le s \le f\} \\ &= \sup\{ c\, I_E(s) \mid 0 \le s \le f\} \\ &= \sup\{ I_E(cs) \mid 0 \le cs \le cf\} \\ &= \int_E (cf) \, dm. \end{aligned}$$

(2) If s, t are two simple non-neg. functions then

$$I_E(s+t) = I_E(s) + I_E(t).$$

Now choose monotone sequences of simple functions with

$0 \le s_1 \le \ldots \le s_n \le \ldots \le f$, $0 \le t_1 \le t_2 \le \ldots \le t_n \le \ldots \le g$

$s_n \to f$, $t_n \to g$. Then

$$0 \le s_1 + t_1 \le s_2 + t_2 \le \ldots \le s_n + t_n \le \ldots \le f + g$$

and $s_n + t_n \longrightarrow f + g$. By Lebesgue monotone convergence Theorem

$$\int_E (f+g) \, dm = \lim_{n\to\infty} \int_E (s_n + t_n) \, dm = \lim_{n\to\infty} \left(\int_E s_n \, dm + \int_E t_n \, dm \right)$$

$$= \lim_{n\to\infty} \int_E s_n \, dm + \lim_{n\to\infty} \int_E t_n \, dm = \int_E f \, dm + \int_E g \, dm.$$

$\square$

Cor 38.2 Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of nonnegative measurable functions. Then $\sum_{n=1}^{\infty} f_n$ is a nonnegative measurable function and for any measurable set E

$$\int_E \left(\sum_{n=1}^{\infty} f_n\right) dm = \sum_{n=1}^{\infty} \int_E f_n \, dm.$$

Proof Let $F_n = \sum_{i=1}^{n} f_i$. Then $0 \le F_1 \le F_2 \le \ldots \le F_n \le \ldots \le \sum_{n=1}^{\infty} f_n$, $\forall n$, F_n is measurable, $\sum_{n=1}^{\infty} f_n = \lim_{n\to\infty} F_n$ is measurable and

$$\int_E \left(\sum_{n=1}^{\infty} f_n\right) dm \overset{\text{Lebesgue monotone conv}}{=} \lim_{n\to\infty} \int_E F_n \, dm \overset{38.2}{=} \lim_{n\to\infty}\left(\sum_{i=1}^{n} \int_E f_i \, dm\right)$$

$$= \sum_{n=1}^{\infty} \left(\int_E f_i \, dm\right).$$

$\square$

Recall $f : \mathbb{R} \to [-\infty, \infty]$ is integrable over E if $\int_E f_+ \, dm$, $\int_E f_- \, dm$ are finite

And then we define

$$\int_E f \, dm := \int_E f_+ \, dm - \int_E f_- \, dm.$$

Lemma 38.3 $f : \mathbb{R} \to [-\infty, \infty]$ is integrable over E $\Leftrightarrow$ $\int_E |f| \, dm$ is finite.

Proof Since $|f| = f_+ + f_-$ and f_+, f_- are measurable $|f|$ is measurable. Moreover by 38.1

$$\int_E |f| \, dm = \int_E f_+ \, dm + \int_E f_- \, dm$$

Since $0 \le \int_E f_+ \, dm$, $\int_E f_- \, dm$

$\int_E |f| \, dm$ is finite $\Leftrightarrow$ $\int_E f_+ \, dm$ and $\int_E f_- \, dm$ are finite.

Notation For a measurable set E

$$L^1(E) = \{ f : E \to [-\infty, \infty] \text{ is measurable and } \int_E |f|\, dm < \infty \}$$

The L^1 norm on $L^1(E)$ is

$$\|f\|_{L^1,E} := \int_E |f|\, dm$$

(Compare: for $x \in \mathbb{R}^n$, $\|x\|_1 = |x_1| + \cdots + |x_n|$)

One can prove that $\| \cdot \|_{L^1,E}$ is a norm and $L^1(E)$ is complete.

Theorem 38.4 Let $E \subseteq \mathbb{R}$ be measurable, $f, g \in L^1(E)$, $c \in \mathbb{R}$. Then

1) $cf \in L^1(E)$ and $\int_E cf\, dm = c(\int_E f\, dm)$

2) $f+g \in L^1(E)$ and

$$\int_E (f+g)\, dm = \int_E f\, dm + \int_E g\, dm$$

Proof If $c \geq 0$, Then $(cf)_+ = c \cdot f_+$, $(cf)_- = c \cdot (f_-)$, so

$$\int_E (cf)\, dm = \int_E cf_+\, dm - \int_E cf_-\, dm$$

$$= c \int_E f_+ dm - c \int_E f_-\, dm = c\left(\int_E f\, dm\right)$$

If $c = -1$, $(cf)_+ = (-f)_+ = f_-$, $(cf)_- = f_+$

and $\int_E (-f)\, dm = \int_E f_-\, dm - \int_E f_+\, dm = (-1) \cdot \int_E f\, dm.$

The result now follows.

(2) Let $h = f+g$. Assume first f, g, h don't change sign on E. We have 6 subcases

1) $f \geq 0$, $g \geq 0$, $h \geq 0$ on E
2) $f \leq 0$, $g \leq 0$ $h \leq 0$ on E
3) $f \geq 0$, $g \leq 0$ $h \geq 0$ on E
4) $f \geq 0$, $g \leq 0$, $h \leq 0$ on E
5) $f \leq 0$, $g \geq 0$, $h \geq 0$ on E
6) $f \leq 0$, $g \geq 0$, $h \leq 0$ on E

If $f, g \geq 0$ then we know $\int_E f\,dm + \int_E g\,dm = \int_E (f+g)\,dm$ by 38.1 (2).

subcase (2) follows from (1) and 38.4 since $\int_E (-h)\,dm = \int_E (-f)\,dm + \int_E (-g)\,dm$

subcase (3) $h = f+g$ is equivalent to $f = h + (-g)$ and then by (1)

$$\int_E f\,dm = \int_E h\,dm + \int_E (-g)\,dm$$ which implies (3)

Similarly subcases (4), (5), (6) reduce to subcase (1).

Now write $E = E_1 \sqcup E_2 \sqcup \cdots \sqcup E_6$

so that $E_i = \{x \in E \mid \text{case } i \text{ holds}\}$

Then $\int_E f\,dm = \sum_{i=1}^{6} \int_{E_i} f\,dm$ (HW 12 #5)

Similar formulas hold for g and h.

$\rightsquigarrow$ the result follows. $\square$

Corollary 38.5 If $f, g \in L^1(E)$ and $f \leq g$ then $\int_E f\,dm \leq \int_E g\,dm$.

Proof $f \leq g \Rightarrow g - f \geq 0$. Hence

$$0 \leq \int_E (g-f)\,dm = \int_E g\,dm + \int_E (-1) f\,dm$$
$$= \int_E g\,dm + (-1)\int_E f\,dm$$

$\Rightarrow \int_E f\,dm \leq \int_E g\,dm$.

Cor 38.6 If $f \in L^1(E)$ then

$$\left| \int_E f\,dm \right| \leq \int_E |f|\,dm$$

Proof Apply monotonicity (38.5): $-f, f \leq |f|$

$\Rightarrow$ $-\int_E f\,dm,\ \int_E f\,dm \leq \int_E |f|\,dm$

hence $\left| \int_E f\,dm \right| \leq \int_E |f|\,dm$.