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<u>Recall</u> A <u>metric space</u> is a pair (E, d) where E is a set $d: E \times E \to [0, \infty)$ a function called <u>metric</u> so that $\forall x, y, z \in E$

1) $d(x,y) = 0 \iff x = y$
2) $d(y,x) = d(x,y)$
3) $d(x,z) \le d(x,y) + d(y,z)$.

An <u>open ball</u> (in) a metric space (E,d) of radius r centered at x is
$$B_r(x) := \{ y \in E \mid d(x,y) < r \}.$$

A subset U of a metric space (E,d) is <u>open</u> if $\forall x \in U$ $\exists r = r(x) > 0$ so that $B_r(x) \subset U$.

<u>Theorem 4.1</u> Let (E,d) be a metric space. Then

i) $\forall$ collection $\{U_i\}_{i \in I}$ of open subsets of E, $\bigcup_{i \in I} U_i$ is open.

ii) $\forall$ finite collection $\{U_1, \ldots, U_k\}$ of open subsets of E $U_1 \cap \ldots \cap U_k$ is open.

iii) $\forall x$, $\forall r > 0$ $B_r(x)$ is open.

<u>Proof</u> (i) Suppose $x \in \bigcup_{i \in I} U_i$. Then $x \in U_j$ for some $j \in I$. Since U_j is open (and $x \in U_j$) $\exists r(x)$ s.t. $B_r(x) \subset U_j \ (\subseteq \bigcup U_i)$
$\Rightarrow \bigcup U_i$ is open.

ii) [really an induction on k]. Suppose $x \in U_1 \cap \ldots \cap U_k$.
Since each U_i is open $\exists r_i > 0$ s.t. $B_{r_i}(x) \subseteq U_i$.
Let $r = \min\{r_1, \ldots, r_k\}$. Then $B_r(x) \subseteq B_{r_i}(x) \subseteq U_i \quad \forall i$
$$\Rightarrow B_r(x) \subseteq \bigcap_{i=1}^{k} U_i.$$

iii) Suppose $y \in B_r(x)$. Then $d(x,y) < r$
$\Rightarrow \delta = r - d(x,y) > 0.$
If $z \in B_\delta(y)$ Then $d(z,y) < \delta$.

$\Rightarrow \; d(x,z) \le d(z,y) + d(y,x) < \delta + d(y,x) = r - d(y,x) + d(y,x) = r$

$\Rightarrow \; z \in B_r(x)$

Since z is arbitrary $B_\delta(y) \subseteq B_r(x)$

Since y is arbitrary $B_r(x)$ is open. $\square$

<u>Note</u> $\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$. So arbitrary intersections of open sets need not be open. (since $\{0\}$ is <u>not</u> open)

<u>Exercise</u> (E,d) metric space

i) Arbitrary intersections of closed sets are closed

ii) <u>Finite</u> unions of closed sets are closed

iii) Closed balls are closed.

<u>Remark</u> An <u>open rectangle</u> in $\mathbb{R}^n$ is a set U of the form

$U = (a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n)$

U is <u>open</u>: $\forall x = (x_1, \ldots, x_n) \in U$ let $r = \frac{1}{2} \cdot \min_{1 \le i \le n} (|a_i - x_i|, |b_i - x_i|)$

Then $B_r(x) \subseteq U$

check that

Similarly $F = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] \subseteq \mathbb{R}^n$, a <u>closed rectangle</u> is closed.

<u>Definition</u> A subset $\emptyset \neq S$ of a metric space (E,d) is <u>bounded</u> if $\exists x \in E, r > 0$ s.t. $B_r(x) \supseteq S$.

<u>Ex</u> In $\mathbb{R}$ $[a,b)$ is bounded: let $x = 0$, $r = |a| + |b|$.

$[0, \infty)$ is <u>not</u> bounded

<u>Ex</u> $E \neq \emptyset$ a set $d(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$. <u>Any</u> subset of E is bounded. !

Theorem 4.2 Suppose $\emptyset \neq S \subseteq \mathbb{R}$ is bounded and closed (w.r.t the standard metric : $d(x,y) = |x-y|$). Then

$$\inf S, \ \sup S \in S$$

i.e. S has the largest and smallest elements.

Proof We argue : $\sup S \in S$ (the proof that $\inf S \in S$ is similar).

Since S is bounded, it's bounded above. [why?]

Hence $L = \sup S$ exists.

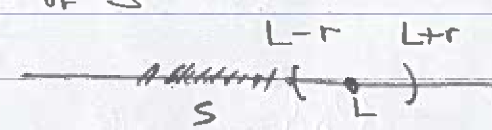
Suppose $L \notin S$. Then $L \in \mathbb{R} \setminus S$, which is open since S is closed. $\Rightarrow \exists\, r > 0$ s.t $\mathbb{R} \setminus S \supseteq B_r(L) = (L-r, L+r)$

Since $(L, \infty) \cap S = \emptyset$, $(L, \infty) \subseteq \mathbb{R} \setminus S$. $\rightarrow$

$\mathbb{R} \setminus S \supseteq (L-r, L+r) \cup (L, \infty) = (L-r, \infty)$.

$\Rightarrow$ $L-r$ is an upper bound of S

Contradiction.



$\therefore$ $L \in S$.

Definition A sequence in a set E is a function

$$s : \mathbb{N} \to E$$

Notation $s = \{s_i\}_{i=1}^{\infty} = \{s_1, s_2, \ldots, s_n, \ldots\}$ or just s_n.

Definition Let (E, d) be a metric space and $s = \{s_i\}_{i=1}^{\infty}$ a sequence. s converges to $L \in E$ if $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ so that $\forall n > N$

$$d(s_n, L) < \varepsilon.$$

If s converges to L we write $s_n \to L$ and say that s is convergent and that L is a limit of s.

Note $s_n \to L \iff \forall \varepsilon > 0 \ \exists N$ s.t. $\{s_n\}_{n > N} \subseteq B_\varepsilon(L)$

$\Leftrightarrow \forall \varepsilon$ all but finitely many s_n's are in $B_\varepsilon(L)$

Note also Since $d(s_n, L) < \varepsilon \Leftrightarrow |d(L, s_n) - 0| < \varepsilon$

$s_n \to L \Leftrightarrow d(s_n, L) \to 0.$

Lemma 4.3 (E, d) metric space, $L \in E$, $\{s_n\}$ a sequence

Then $s_n \to L \Leftrightarrow \forall$ open set U with $L \in U$ $\exists N$ s.t. $s_n \in U$ for $n > N$.

Proof ($\Rightarrow$) Suppose $s_n \to L$, U open, $L \in U$. Then $\exists r > 0$ s.t.

$B_r(L) \subseteq U$. Since $s_n \to L$ $\exists N$ s.t. for $n > N$ $s_n \in B_r(L) \subseteq U$.

($\Leftarrow$) Suppose $\forall U$ open w. $L \in U$ $\exists N$ s.t. $n > N \Rightarrow s_n \in U$.

and $\varepsilon > 0$. Then $B_\varepsilon(L)$ is open and contains L.

$\Rightarrow \exists N$ s.t. for $n > N$ $s_n \in B_\varepsilon(L) \Rightarrow$ for $n > N$

$d(L, s_n) < \varepsilon$. $\square$

Ex $\{\frac{1}{n}\}_{n \geq 1} \subseteq \mathbb{R}$ converges to 0.

Proof $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ s.t. $\frac{1}{\varepsilon} < N$. Then for $n > N$

$\frac{1}{\varepsilon} < n \Rightarrow 0 < \frac{1}{n} < \varepsilon \Rightarrow |0 - \frac{1}{n}| < \varepsilon.$

Ex $(\frac{1}{10^n})_{n \geq 1}$ converges to 0.

Reason/proof $\forall n$ $n < 10^n$. Given $\varepsilon > 0$ $\exists N \in \mathbb{N}$

s.t. $\varepsilon > \frac{1}{n} > \frac{1}{10^n}$ for $n > N$. $\Rightarrow |0 - \frac{1}{10^n}| < \varepsilon$ for $n > N$

Lemma 4.4 Convergent sequences are bounded

Proof Suppose we have (E, d) and $s_n \to L$. Then $\exists N$ s.t.

for $n > N$ $d(L, s_n) < 1$

Let $r = \max\{1, d(L, s_1), \ldots d(L, s_N)\}$.

Then $\forall k$ $d(s_k, L) < r \Rightarrow s_k \in B_r(L)$ $\forall k$.

$\square$