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Fr.

Recall A sequence $\{s_n\}$ in a metric space (E,d) converges to $L \in E$ if $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ so that

$$n > N \Rightarrow d(s_n, L) < \varepsilon.$$

Lemma 4.3 (didn't prove it last time) A sequence $\{s_n\}$ in (E,d) converges to $L \iff \forall$ open set $U \subseteq E$ with $L \in U$

$$\exists N \in \mathbb{N} \text{ so that } n > N \Rightarrow s_n \in U.$$

Proof ($\Rightarrow$) Suppose $s_n \to L$, $U \subseteq E$ is open, $L \in U$.
Then $\exists \varepsilon > 0$ st $B_\varepsilon(L) \subseteq U$. Since $s_n \to L$ $\exists N$ st
$n > N \Rightarrow d(s_n, L) < \varepsilon$, ie. $s_n \in B_\varepsilon(L) \subseteq U$.

($\Leftarrow$) $\forall \varepsilon > 0$, $B_\varepsilon(L)$ is open and contains L.
Therefore $\exists N$ st $n > N \Rightarrow s_n \in B_\varepsilon(L)$, ie. $d(s_n, L) < \varepsilon$ for $n > N$. □

Lemma 4.4 Convergent sequences are bounded.

Proof Suppose $s_n \to L$ in a metric space (E,d).
Then $\exists N$ st $n > N \Rightarrow d(s_n, L) < 1$
Let $r = \max\{1, d(s_1, L), \dots \quad d(s_N, L)\}$
Then $\forall r \quad d(s_k, L) < r + 1 \Rightarrow s_k \in B_{r+1}(L) \quad \forall k$.

Fact Suppose $\{s_n\}$ is a sequence in a metric space (E,d).
If $s_n \to L_1$ and $s_n \to L_2$ then $L_1 = L_2$.

Proof see p.46 of text.

Def Let $s : \mathbb{N} \to E$ be a sequence. A subsequence of s is a function $f : \mathbb{N} \to E$ of the form $f = s \circ n$ where $n : \mathbb{N} \to \mathbb{N}$ is a strictly increasing function.
ie. n is a sequence $n_1, n_2, \dots n_k \dots$ of natural numbers with

$$1 \le n_1 < n_2 < n_3 \dots < n_k < n_{k+1} \dots$$

and $s \circ n = (s_{n_1}, s_{n_2}, \dots \dots)$ Note $n_k \ge k$ $\forall k$
Why?

Fact If $s_n \to L$ and $\{s_{n_k}\}_{k=1}^{\infty}$ is a subsequence then $s_{n_k} \to L$.

Proof see p 46.

Recall A subset C of a metric space (E,d) is closed iff $E \setminus C$ is open : $\forall\, x \notin C\ \exists\, r>0$ s.t. $B_r(x) \cap C = \emptyset$.

Lemma 5.1 Let C be a closed subset of (E,d), $\{s_n\}$ a sequence in C (ie $s_n \in C\ \forall n$). If $s_n \to L$ then $L \in C$.
Conversely if $\forall$ convergent sequence $\{s_n\}$ in C $\lim s_n \in C$ then C is closed.

Proof ($\Rightarrow$) Suppose C is closed, $\{s_n\} \in C$ and $s_n \to L$.
If $L \notin C$ then $\exists\, r>0$ s.t. $B_r(L) \subseteq E \setminus C$ (since $E \setminus C$ is open)
Since $s_n \to L$ $\exists\, N \in \mathbb{N}$ s.t. $s_n \in B_r(L)$ for $n > N$.
Contradiction since $s_n \in C\ \forall n$ and $B_r(L) \cap C = \emptyset$.

($\Leftarrow$) Suppose C is not closed. Then $E \setminus C$ is not open.
$\Rightarrow \exists\, p \in E \setminus C$ so that $\forall\, r>0$ $B_r(p) \not\subseteq E \setminus C$ ie
$B_r(p) \cap C \neq \emptyset$.
In particular $\forall\, n \in \mathbb{N}$ $\exists\, x_n \in B_{1/n}(p) \cap C$
Then $\{x_n\}$ is a sequence with $d(x_n, p) < \frac{1}{n}$.
Given $\varepsilon > 0$ $\exists\, N$ s.t. $\frac{1}{N} < \varepsilon$. Then for $n > N$,
$$\varepsilon > \frac{1}{N} > \frac{1}{n} > d(x_n, p)$$
$$\Rightarrow \quad x_n \to p \notin C$$
We proved: if C is not closed $\exists$ sequence $\{x_n\}$ in C with $x_n \to p \notin C$.

$\square$

Sequences in $\mathbb{R}$

Proposition 5.2 Let $\{a_n\}$, $\{b_n\}$ be sequences in $\mathbb{R}$ with $a_n \to a$, $b_n \to b$. Then

i) $a_n + b_n \to a + b$

ii) $a_n b_n \to ab$

iii) $c a_n \to c a$ $\forall c \in \mathbb{R}$

iv) if $b \neq 0$ and if $b_n \neq 0$ $\forall n$, Then $\frac{a_n}{b_n} \to \frac{a}{b}$

v) if $a_n \leq b_n$ $\forall n$ then $a \leq b$.

Proof (i) - (ii) see pp. 48-49.

(iii) Note For a constant sequence $n \mapsto c$ $\forall n$, $c \to c$. So (ii) $\Rightarrow$ (iii).

(iv) It is enough to prove: $\frac{1}{b_n} \to \frac{1}{b}$ for Then, by (ii)

$$\frac{a_n}{b_n} = a_n \cdot \frac{1}{b_n} \to a \cdot \frac{1}{b}.$$

Now,

$$\left| \frac{1}{b_n} - \frac{1}{b} \right| = \frac{|b - b_n|}{|b_n||b|}.$$

Given $\varepsilon > 0$ let

$$\varepsilon' = \min\left(\frac{|b|}{2}, \frac{|b|^2}{2} \varepsilon \right)$$

Then, since $b_n \to b$, $\exists N$ with $n > N \Rightarrow |b_n - b| < \varepsilon'$.

, Since $|b| = |b - b_n + b_n| \leq |b_n - b| + |b_n|$, $|b_n| \geq |b| - |b_n - b|$

And then, for $n > N$

$$|b_n| > |b| - \varepsilon' \geq |b| - \frac{|b|}{2} = |b|/2.$$

Therefore, for $n > N$ $\left| \frac{1}{b_n} - \frac{1}{b} \right| < \frac{|b - b_n|}{|b|} \cdot \frac{2}{|b|} < \frac{\frac{|b|^2}{2} \varepsilon \cdot 2}{|b|^2} = \varepsilon$

$\therefore$ $\frac{1}{b_n} \to \frac{1}{b}$.

(v) $a_n \leq b_n \Leftrightarrow b_n - a_n \geq 0 \Leftrightarrow b_n - a_n \in [0, \infty)$ $\forall n$

$b - a = \lim (b_n - a_n) \in [0, \infty)$ since $[0, \infty)$ is closed. $\square$

Interior, closure, boundary

Fix a metric space (E,d) and a subset $S \subseteq E$. We define

Interior $(S) \equiv S^\circ := \bigcup_{\substack{O \subseteq S \\ O \text{ open}}} O$ = largest open set contained in S (it may be $\emptyset$)

Closure $(S) \equiv \bar{S} = \bigcap_{\substack{C \text{ closed} \\ S \subseteq C}} C$ smallest closed set containing S

Boundary$(S) \equiv \partial S = \bar{S} \setminus S^\circ$

Exterior $(S) \equiv \mathrm{Ext}(S) :=$ interior of $E \setminus S$

Ex $E = \mathbb{R}$, standard metric, $S = \mathbb{Q}$ the rationals. Then

$\mathbb{Q}^\circ = \emptyset$ since $\forall q \in \mathbb{Q}\ \forall r > 0$ $B_r(q) = (q-r, q+r) \cap (\mathbb{R} \setminus \mathbb{Q}) \neq \emptyset$

So $\nexists r$ s.t. $B_r(q) \subseteq \mathbb{Q}$.

$\bar{\mathbb{Q}} = \mathbb{R}$

$\Rightarrow \partial \mathbb{Q} = \bar{\mathbb{Q}} \setminus \mathbb{Q}^\circ = \mathbb{R}$.

Ex $E = \{-1, 0, 1\} \subseteq \mathbb{R}$, $d(x,y) = |x-y|$ $\forall x,y \in E$.

$B_1(0) = \{0\}$. Similarly $B_1(1) = \{1\}$, $B_1(-1) = \{-1\}$.

$\Rightarrow$ every subset of E is open (and closed).

$\Rightarrow$ closure $(B_1(0)) = \{0\}$.

Note $\bar{B}_1(0) = \{y \in E \mid |0-y| \leq 1\} = E$

So closed ball $\bar{B}_1(0) \neq$ closure of $B_1(0) = \overline{B_1(0)}$

Ex $S = \{\frac{1}{n} \mid n > 0\} \subseteq \mathbb{R}$

$S^\circ = \emptyset$, $\bar{S} = \{0\} \cup S$

$\partial S = \bar{S} \setminus S^\circ = \{0\} \cup S$

$\mathrm{Ext}(S) = (\mathbb{R} \setminus S)^\circ = \mathbb{R} \setminus (S \cup \{0\})$

since every open ball about $\{0\}$ contains points of S