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6.1

Recall $S \subseteq (E,d)$, a subset of a metric space

interior of $S \equiv S^0 := \bigcup_{\substack{U \subseteq S \\ U \text{ open}}} U$

closure of $S \equiv \bar{S} := \bigcap_{\substack{C \supseteq S \\ C \text{ closed}}} C$

boundary of $S \equiv \partial S = \bar{S} \setminus S^0$

$\rightarrow$ exterior of $S \equiv \mathrm{Ext}(S) :=$ interior of $E \setminus S = (E \setminus S)^0$

Ex $S = \{\frac{1}{n} \mid n \in \mathbb{N}\} \subseteq \mathbb{R}$ $S^0 = \emptyset$ $\bar{S} = \{0\} \cup S$ (since $\frac{1}{n} \to 0$, $0 \in \bar{S}$)

$\partial S = \bar{S} \setminus S^0 = \{0\} \cup S$

$\mathrm{Ext}(S) = (\mathbb{R} \setminus S)^0 = \mathbb{R} \setminus (S \cup \{0\})$

since every open ball around 0 contains points of S hence is not in $\mathbb{R} \setminus S$.

Theorem 6.1 Let (E,d) be a metric space, $S \subseteq E$. Then

(i) $S^0 = \{x \in S \mid \exists \varepsilon > 0 \text{ s.t. } B_\varepsilon(x) \subseteq S\}$

(ii) $E \setminus \bar{S} = (E \setminus S)^0$

(iii) $\bar{S} = \{x \in E \mid \exists \text{ a sequence } \{s_n\} \subseteq S \text{ s.t. } s_n \to x\}$.

(iv) $\partial S = (E \setminus S^0) \cap (E \setminus ((E \setminus S)^0))$

(v) E is the disjoint union of S^0, ∂S and $\mathrm{Ext}(S) \equiv (E \setminus S)^0$:

$$E = S^0 \sqcup \partial S \sqcup (E \setminus S)^0$$

Proof

(ii) $E \setminus \bar{S} = E \setminus \bigcap_{\substack{C \text{ closed} \\ S \subseteq C}} C = \bigcup_{\substack{C \text{ closed} \\ E \setminus S \supseteq E \setminus C}} (E \setminus C) = \bigcup_{\substack{O \text{ open} \\ E \setminus S \supseteq O}} O = (E \setminus S)^0$

(i) Suppose $x \in S^0$. Then, since S^0 is open, $\exists r > 0$ s.t. $B_r(x) \subseteq S^0 \subseteq S$

$\Rightarrow x \in \{x' \in S \mid \exists \varepsilon > 0 \text{ with } B_\varepsilon(x') \subseteq S\}$.

Suppose $x \in S$ and $\exists \varepsilon > 0$ s.t. $B_\varepsilon(x) \subseteq S$. Then $B_\varepsilon(x) \subseteq \bigcup_{\substack{O \text{ open} \\ O \subseteq S}} O = S^0$.

(iii) Suppose $x \in \bar{S}$. Since $\bar{S} = (E \setminus S)^c$, $x \notin E \setminus \bar{S} = (E \setminus S)^\circ$

$\Rightarrow \forall r \quad B_r(x) \not\subset E \setminus S \Rightarrow \forall r \quad B_r(x) \cap S \neq \emptyset$.

Thus $\forall n > 0 \ \exists s_n \in B_{1/n} \cap S$. And then, $s_n \to x$.

Conversely if $\{s_n\} \subset S$ is a sequence and $s_n \to x$

Then $\{s_n\} \subset \bar{S}$ and $x \in \bar{S}$ since $\bar{S}$ is closed.

(iv) $$\partial S = \bar{S} \setminus S^\circ = \bar{S} \cap (E \setminus S^\circ) = (E \setminus (E \setminus \bar{S})) \cap (E \setminus S^\circ)$$
$$\overset{(ii)}{=} (E \setminus (E \setminus S)^\circ) \cap (E \setminus S^\circ)$$
$$= (E \setminus \mathrm{Ext}(S)) \cap (E \setminus S^\circ)$$

(v) $E = (E \setminus \bar{S}) \amalg \bar{S} \overset{(ii)}{=} (E \setminus S)^\circ \amalg (\partial S \amalg S^\circ)$

□

Monotone sequences in $\mathbb{R}$

Definition

A sequence $\{a_n\}$ in $\mathbb{R}$ is increasing if $a_1 \le a_2 \le a_3 \le \dots \le a_n \le \dots$

Eg $a_n = 1 \ \forall n$ is increasing. So is $a_n = 1 - \frac{1}{n}$.

Def A sequence $\{b_n\}$ in $\mathbb{R}$ is decreasing if
$$b_1 \ge b_2 \ge \dots \ge b_n \ge \dots$$

Ex $b_n = 1$ is decreasing; $b_n = -n$ is also decreasing.

Def A sequence $\{a_n\}$ is monotone if it's either increasing or if it's decreasing.

Theorem 6.2 Any bounded monotone sequence in $\mathbb{R}$ converges.

Proof (for increasing sequences)

Suppose $\{a_n\}$ is increasing and bounded.

Then $S = \{a_n \mid n \in \mathbb{N}\}$ is bounded above.

Since $\mathbb{R}$ is complete, $L := \sup S$ exists.
We argue: $a_n \to L$.
Given $\varepsilon > 0$, $L - \varepsilon$ is not an upper bound of S.
$\Rightarrow \exists N$ s.t. $L - \varepsilon < a_N$.
But then for $n \geq N$ $\quad L - \varepsilon < a_N \leq a_n < L < L + \varepsilon$
$\Rightarrow a_n \to L$.

The proof in the case $\{a_n\}$ is decreasing is similar:
$$a_n \to \inf \{a_n \mid n \in \mathbb{N}\}$$

Example Define a sequence $\{a_n\}$ by $a_1 = \sqrt{2}$, $a_2 = \sqrt{2+\sqrt{2}}$ ...
$$a_n = \sqrt{2 + a_{n-1}}$$
Show that $\{a_n\}$ converges and compute $\lim a_n$.

Solution (i) We argue that $\sqrt{2} \leq a_n \leq 2$ for all n.
Induction on n: $n = 1$ $\quad \sqrt{2} = a_1 \leq 2$.
Suppose $\sqrt{2} \leq a_{n-1} \leq 2$. Then $a_n = \sqrt{2 + a_{n-1}} \geq \sqrt{2+\sqrt{2}} > \sqrt{2}$
and $a_n = \sqrt{2 + a_{n-1}} \leq \sqrt{2+2} = 2$.

(ii) $\{a_n\}$ is increasing:
$$a_n - a_{n-1} = \sqrt{2+a_{n-1}} - a_{n-1} = \frac{\left(\sqrt{2+a_{n-1}} - a_{n-1}\right)\left(\sqrt{2+a_{n-1}} + a_{n-1}\right)}{\sqrt{2+a_{n-1}} + a_{n-1}}$$
$$= \frac{\left(\sqrt{2+a_{n-1}}\right)^2 - a_{n-1}^2}{\sqrt{2+a_{n-1}} + a_{n-1}} = \frac{2 + a_{n-1} - a_{n-1}^2}{\text{positive}} = \frac{(2 - a_{n-1})(a_{n-1} + 1)}{\text{positive}} \geq 0$$
since $2 - a_{n-1} \geq 0$ and $a_{n-1} + 1 \geq \sqrt{2} + 1 > 0$.

By Thm 6.2 $L = \lim a_n$ exists.
Since $a_n = \sqrt{2 + a_{n-1}}$, $\quad a_n^2 = 2 + a_{n-1}$
Hence $\quad L^2 = (\lim a_n)^2 = \lim (a_n^2) = \lim (2 + a_{n-1}) = 2 + L$.
$\Rightarrow L = 2$ or $L = -1$

Since $a_n \geq \sqrt{2}$, $\quad \lim a_n \geq \sqrt{2}$. $\quad \therefore \; L = 2$.

__Definition__ A sequence $\{a_n\} \subseteq \mathbb{R}$ diverges to $+\infty$ if $\forall M \in \mathbb{R}$ $\exists N \in \mathbb{N}$ so that $a_n > M$ for $n > N$.
$\{a_n\}$ diverges to $-\infty$ if $\{-a_n\}$ diverges to $+\infty$, ie $\forall M \in \mathbb{R}$ $\exists N \in \mathbb{N}$ so that $a_n < M$ for $n > N$.
__Notation:__ $a_n \to +\infty$, $a_n \to -\infty$.

__Thm 6.3__ A monotone sequence in $\mathbb{R}$ either converges or diverges to $+\infty$ or diverges to $-\infty$.

__Proof__ If a sequence is increasing and __not__ bounded, it diverges to $+\infty$.
If a sequence is decreasing and not bounded it diverges to $-\infty$.

Lim sup and lim inf.

__Note:__ Suppose $S, T \subseteq \mathbb{R}$ are bounded and $T \subseteq S$. Then $\sup T \le \sup S$ and $\inf S \le \inf T$.

Cl. Suppose $\{s_n\}$ is a sequence in $\mathbb{R}$, bounded above. Then $v_N := \sup\{s_n \mid n \ge N\}$ exists and, since

$$\{s_n \mid n \ge N\} \supseteq \{s_n \mid n \ge N+1\}$$

$$v_{N+1} \le v_N \quad \forall N \Rightarrow \{v_N\} \text{ is monotone}$$

Hence either $\{v_N\}_{N \in \mathbb{N}}$ converges or diverges to $-\infty$.
We define

$$\limsup s_n := \lim v_N = \lim_{N \to \infty} (\sup\{s_n \mid n \ge N\})$$

(which may be a number or $-\infty$)
Similarly, if $\{s_n\}$ is __bounded below__

$$\liminf s_n = \lim_{N \to \infty} (\inf\{s_n \mid n \ge N\})$$

a number or $+\infty$.